CHAPTER

5 Discovering and Proving Polygon Properties

The mathematicians may well nod their heads in a friendly and interested manner—I still am a tinkerer to them. And the "artistic" ones are primarily irritated. Still, maybe I'm on the right track if I experience more joy from my own little images than from the most beautiful camera in the world . . ."

Still Life and Street, M. C. Escher, 1967–1968
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OBJECTIVES

In this chapter you will

- study properties of polygons
- discover relationships among their angles, sides, and diagonals
- learn about real-world applications of special polygons

LESSON

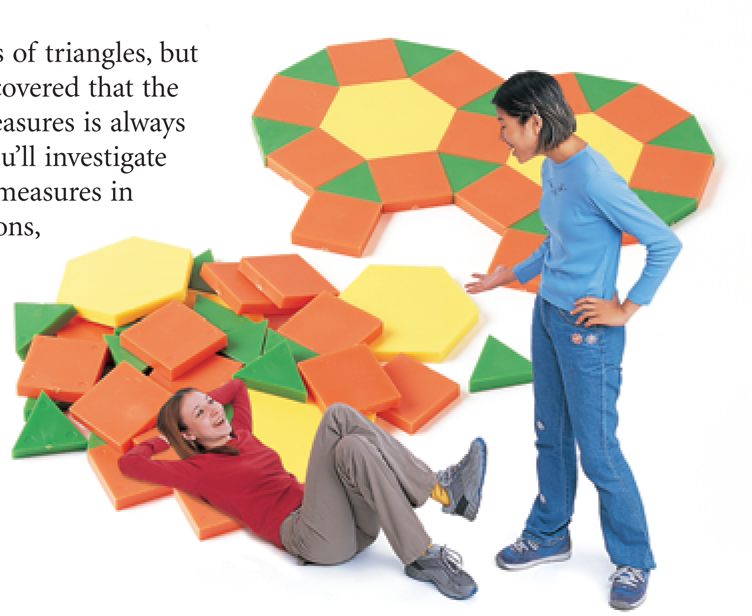
5.1

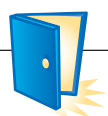
Polygon Sum Conjecture

I find that the harder I work, the more luck I seem to have.

THOMAS JEFFERSON

There are many kinds of triangles, but in Chapter 4, you discovered that the sum of their angle measures is always 180°. In this lesson you'll investigate the sum of the angle measures in quadrilaterals, pentagons, and other polygons. Then you'll look for a pattern in the sum of the angle measures in *any* polygon.

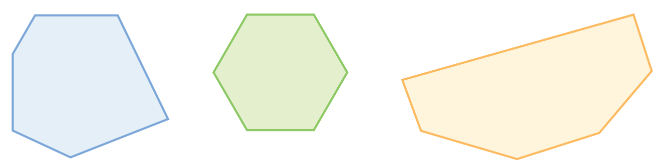




Investigation

Is There a Polygon Sum Formula?

For this investigation each person in your group should draw a different version of the same polygon. For example, if your group is investigating hexagons, try to think of different ways you could draw a hexagon.



Step 1 Draw the polygon. Carefully measure all the interior angles, then find the sum.

Step 2 Share your results with your group. If you measured carefully, you should all have the same sum! If your answers aren't exactly the same, find the average.

Step 3 Copy the table below. Repeat Steps 1 and 2 with different polygons, or share results with other groups. Complete the table.

Number of sides of polygon	3	4	5	6	7	8	…	n
Sum of measures of angles	180°						…	

You can now make some conjectures.

Quadrilateral Sum Conjecture

C-30

The sum of the measures of the four angles of any quadrilateral is ?.

Pentagon Sum Conjecture

C-31

The sum of the measures of the five angles of any pentagon is ?.

If a polygon has n sides, it is called an ***n*-gon.**

Step 4 Look for a pattern in the completed table. Write a general formula for the sum of the angle measures of a polygon in terms of the number of sides, n.

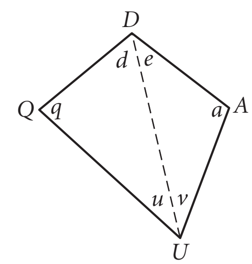
Polygon Sum Conjecture

C-32

The sum of the measures of the n interior angles of an n-gon is _?_.

You used inductive reasoning to discover the formula. Now you can use deductive reasoning to see why the formula works.

Step 5 Draw all the diagonals from *one* vertex of your polygon. How many triangles do the diagonals create? How does the number of triangles relate to the formula you found? How can you check that your formula is correct for a polygon with 12 sides?



Step 6 Write a short paragraph proof of the Quadrilateral Sum Conjecture. Use the diagram of quadrilateral $QUAD$. (Hint: Use the Triangle Sum Conjecture.)

EXERCISES

You will need

Geometry software for Exercise 19

1. Use the Polygon Sum Conjecture to complete the table.

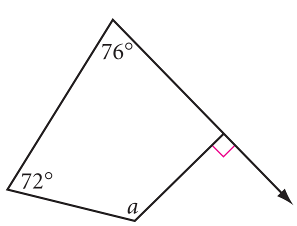
Number of sides of polygon	7	8	9	10	11	20	55	100
Sum of measures of angles								

2. What is the measure of each angle of an equiangular pentagon? An equiangular hexagon? Complete the table. ⓗ

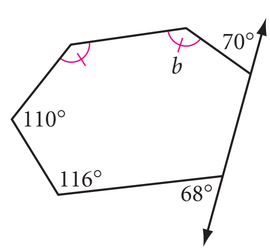
Number of sides of equiangular polygon	5	6	7	8	9	10	12	16	100
Measures of each angle of equiangular polygon									

In Exercises 3–8, use your conjectures to calculate the measure of each lettered angle.

3. $a =$ _?_

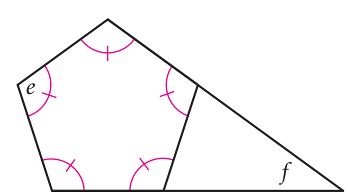


4. $b =$ _?_



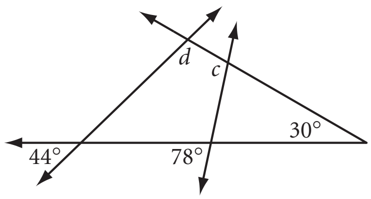
5. $e =$ _?_

$f =$ _?_



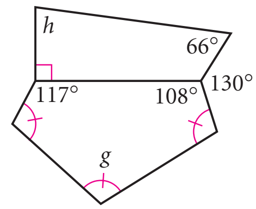
6. $c = \underline{\ ?\ }$

$d = \underline{\ ?\ }$ ⓗ



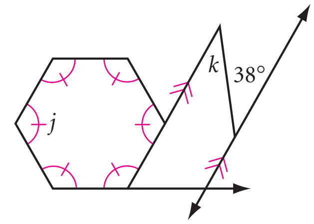
7. $g = \underline{\ ?\ }$ ⓗ

$h = \underline{\ ?\ }$

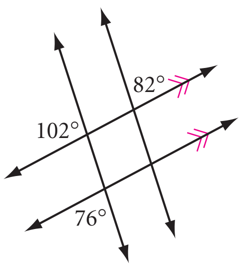


8. $j = \underline{\ ?\ }$

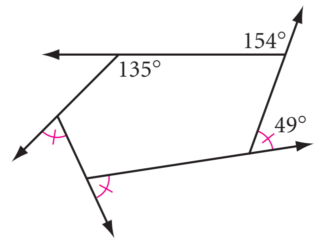
$k = \underline{\ ?\ }$



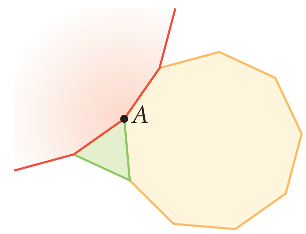
9. What's wrong with this picture?



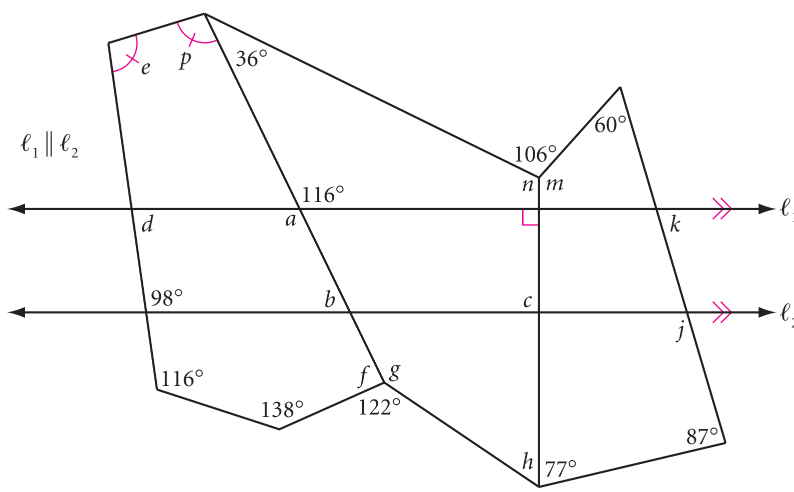
10. What's wrong with this picture?



11. Three regular polygons meet at point A. How many sides does the largest polygon have?



12. Trace the figure at right. Calculate each lettered angle measure.



13. How many sides does a polygon have if the sum of its angle measures is 2700°? ⓗ

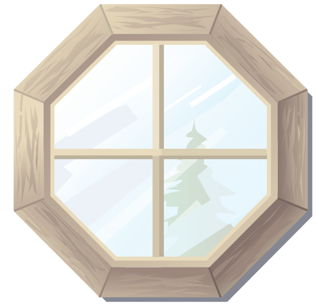
14. How many sides does an equiangular polygon have if each interior angle measures 156°? ⓗ

15. Archaeologist Ertha Diggs has uncovered a piece of a ceramic plate. She measures it and finds that each side has the same length and each angle has the same measure.

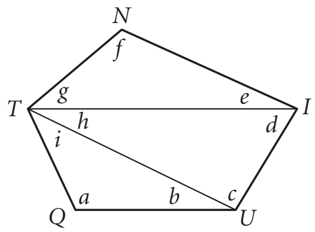
She conjectures that the original plate was the shape of a regular polygon. She knows that if the original plate was a regular 16-gon, it was probably a ceremonial dish from the third century. If it was a regular 18-gon, it was probably a palace dinner plate from the twelfth century.

If each angle measures 160°, from what century did the plate likely originate?

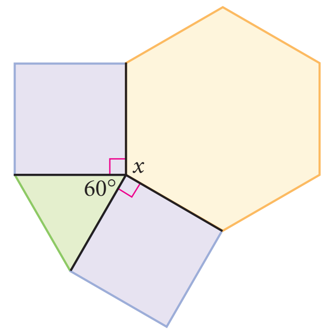
16. **APPLICATION** You need to build a window frame for an octagonal window like this one. To make the frame, you'll cut identical trapezoidal pieces. What are the measures of the angles of the trapezoids? Explain how you found these measures.



17. Use this diagram to prove the Pentagon Sum Conjecture. ⓗ



Review

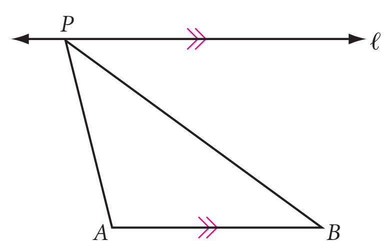


18. This figure is a detail of one vertex of the tiling at the beginning of this lesson. Find the missing angle measure x.

19. ***Technology*** Use geometry software to construct a quadrilateral and locate the midpoints of its four sides. Construct segments connecting the midpoints of opposite sides. Construct the point of intersection of the two segments. Drag a vertex or a side so that the quadrilateral becomes concave. Observe these segments and make a conjecture.

20. Write the equation of the perpendicular bisector of the segment with endpoints $(-12, 15)$ and $(4, -3)$.

21. $\triangle ABC$ has vertices $A(0, 0)$, $B(-4, -2)$, and $C(8, -8)$. What is the equation of the median to side $\overline{AB}$?



22. Line ℓ is parallel to $\overleftrightarrow{AB}$. As P moves to the right along ℓ, which of these measures will always increase?

A. The distance PA

B. The measure of $\angle APB$

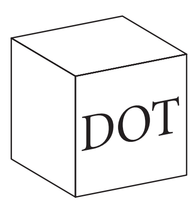
C. The perimeter of $\triangle ABP$

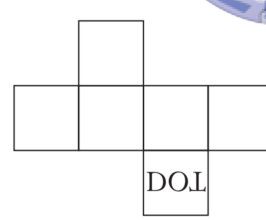
D. The measure of $\angle ABP$

IMPROVING YOUR VISUAL THINKING SKILLS

Net Puzzle

The clear cube shown has the letters *DOT* printed on one face. When a light is shined on that face, the image of *DOT* appears on the opposite face. The image of *DOT* on the opposite face is then painted. Copy the net of the cube and sketch the painted image of the word, *DOT*, on the correct square and in the correct position.





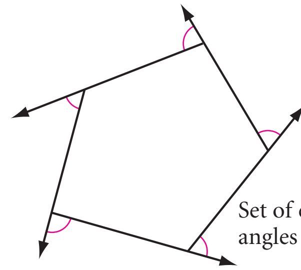
LESSON 5.2

Exterior Angles of a Polygon

If someone had told me I would be Pope someday, I would have studied harder.

POPE JOHN PAUL I

In Lesson 5.1, you discovered a formula for the sum of the measures of the *interior* angles of any polygon. In this lesson you will discover a formula for the sum of the measures of the *exterior* angles of a polygon.



Best known for her participation in the Dada Movement, German artist Hannah Hoch (1889–1978) painted *Emerging Order* in the Cubist style. Do you see any examples of exterior angles in the painting?

Investigation

Is There an Exterior Angle Sum?

You will need

- a straightedge
- a protractor

Let's use some inductive and deductive reasoning to find the exterior angle measures in a polygon.

Each person in your group should draw the same kind of polygon for Steps 1–5.

Step 1 Draw a large polygon. Extend its sides to form a set of exterior angles.

Step 2 Measure all the *interior* angles of the polygon except one. Use the Polygon Sum Conjecture to calculate the measure of the remaining interior angle. Check your answer using your protractor.

Step 3 Use the Linear Pair Conjecture to calculate the measure of each exterior angle.

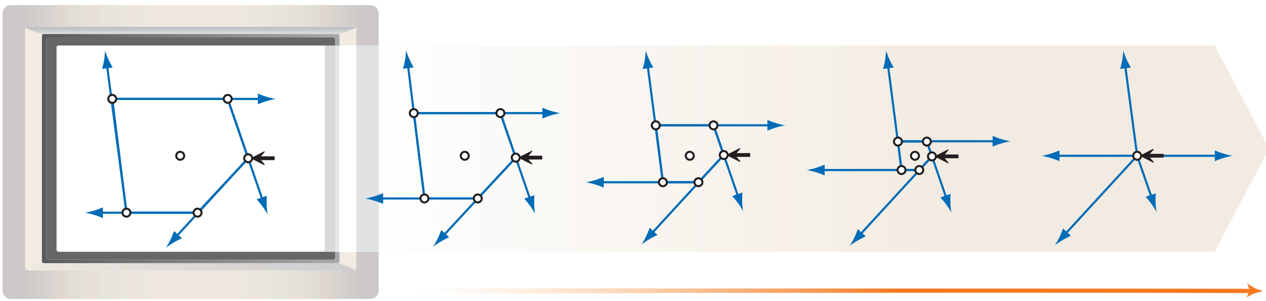
Step 4 Calculate the sum of the measures of the exterior angles. Share your results with your group members.

Step 5 Repeat Steps 1–4 with different kinds of polygons, or share results with other groups. Make a table to keep track of the number of sides and the sum of the exterior angle measures for each kind of polygon. Find a formula for the sum of the measures of a polygon's exterior angles.

C-33

Exterior Angle Sum Conjecture

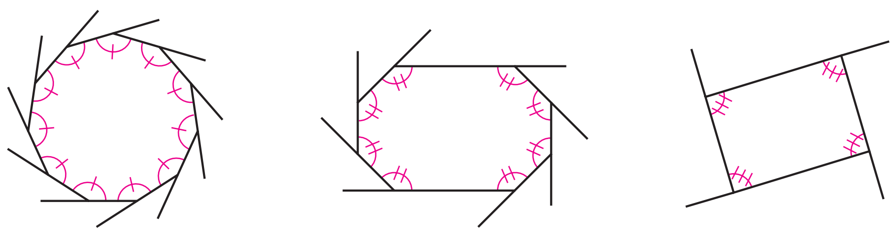
For any polygon, the sum of the measures of a set of exterior angles is __?__.



Step 6 Study the software construction above. Explain how it demonstrates the Exterior Angle Sum Conjecture. For an interactive version of this sketch, visit www.keymath.com/DG .

Step 7 Using the Polygon Sum Conjecture, write a formula for the measure of each interior angle in an equiangular polygon.

Step 8 Using the Exterior Angle Sum Conjecture, write the formula for the measure of each exterior angle in an equiangular polygon.



Step 9 Using your results from Step 8, you can write the formula for an interior angle a different way. How do you find the measure of an interior angle if you know the measure of its exterior angle? Complete the next conjecture.

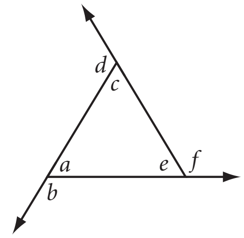
C-34

Equiangular Polygon Conjecture

You can find the measure of each interior angle of an equiangular n-gon by using either of these formulas: __?__ or __?__.

EXERCISES

1. Complete this flowchart proof of the Exterior Angle Sum Conjecture for a triangle.



Flowchart Proof

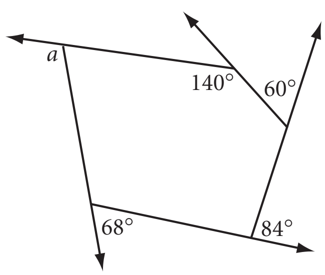
1. $a + b = 180°$ — $\underline{\ ?\ }$
2. $c + d = 180°$ — $\underline{\ ?\ }$
3. $e + f = 180°$ — $\underline{\ ?\ }$
4. $a + b + c + d + e + f = \underline{\ ?\ }°$ — Addition property of equality
5. $a + c + e = \underline{\ ?\ }°$ — $\underline{\ ?\ }$
6. $b + d + f = \underline{\ ?\ }°$ — Subtraction property of equality

2. What is the sum of the measures of the exterior angles of a decagon?

3. What is the measure of an exterior angle of an equiangular pentagon? An equiangular hexagon?

In Exercises 4–9, use your new conjectures to calculate the measure of each lettered angle.

4.



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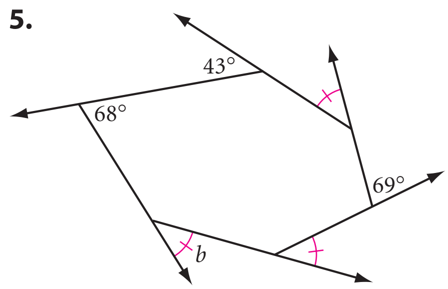
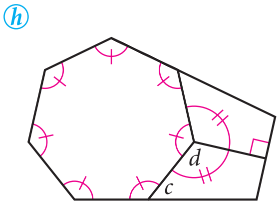


6. ⓗ



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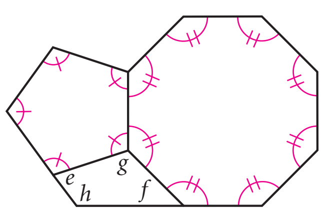

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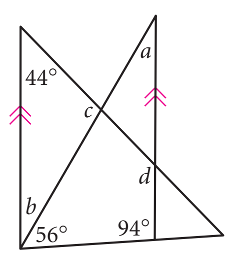

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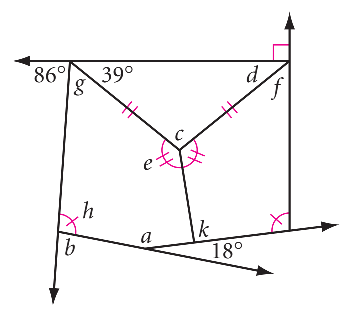


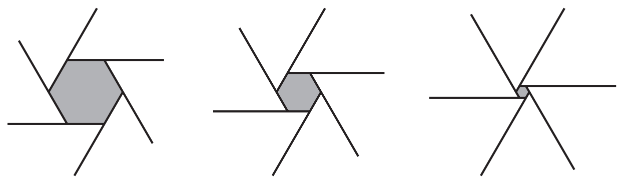
10. How many sides does a regular polygon have if each exterior angle measures 24°? ⓗ

11. How many sides does a polygon have if the sum of its interior angle measures is 7380°?

12. Is there a maximum number of obtuse exterior angles that any polygon can have? If so, what is the maximum? If not, why not? Is there a minimum number of acute interior angles that any polygon must have? If so, what is the minimum? If not, why not? ⓗ

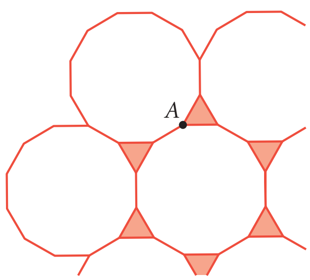
Technology CONNECTION

The aperture of a camera is an opening shaped like a regular polygon surrounded by thin sheets that form a set of exterior angles. These sheets move together or apart to close or open the aperture, limiting the amount of light passing through the camera's lens. How does the sequence of closing apertures shown below demonstrate the Exterior Angle Sum Conjecture? Does the number of sides make a difference in the opening and closing of the aperture?

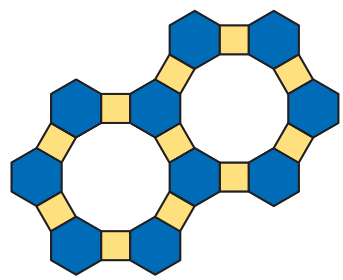


Review

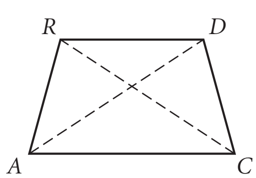
13. Name the regular polygons that appear in the tiling below. Find the measures of the angles that surround point A in the tiling.



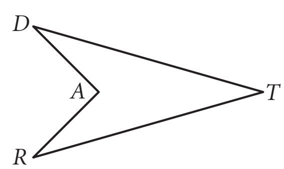
14. Name the regular polygons that appear in the tiling below. Find the measures of the angles that surround any vertex point in the tiling.



15. $\angle RAC \cong \angle DCA$, $\overline{CD} \cong \overline{AR}$, $\overline{AC} \parallel \overline{DR}$. Is $\overline{AD} \cong \overline{CR}$? Why? ⓗ



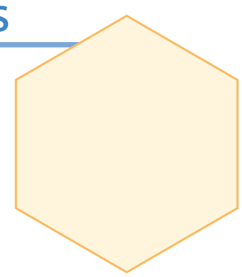
16. $\overline{DT} \cong \overline{RT}$, $\overline{DA} \cong \overline{RA}$. Is $\angle D \cong \angle R$? Why? ⓗ

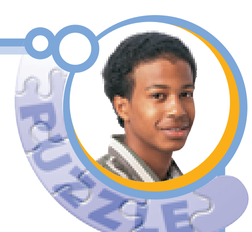


IMPROVING YOUR VISUAL THINKING SKILLS

Dissecting a Hexagon II

Make six copies of the hexagon at right by tracing it onto your paper. Then divide each hexagon into twelve identical parts in a different way.





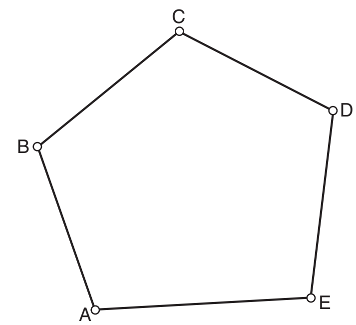
THE GEOMETER'S SKETCHPAD

Exploration

Star Polygons

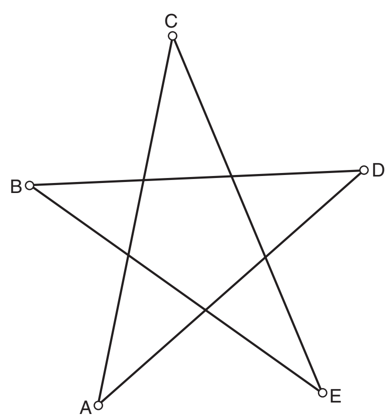
If you arrange a set of points roughly around a circle or an oval, and then you connect each point to the next with segments, you should get a convex polygon like the one at right. What do you get if you connect every second point with segments? You get a star polygon like the ones shown in the activity below.

In this activity, you'll investigate the angle measure sums of star polygons.

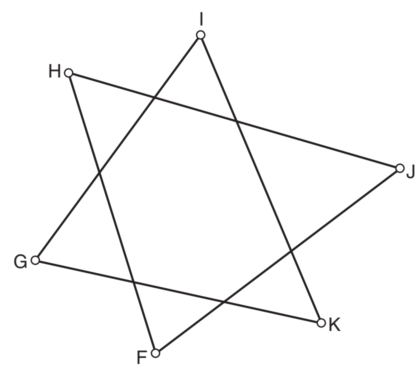


Activity

Exploring Star Polygons



5-pointed star *ABCDE*



6-pointed star *FGHIJK*

Step 1 Draw five points A through E in a circular path, clockwise.

Step 2 Connect every second point with $\overline{AC}$, $\overline{CE}$, $\overline{EB}$, $\overline{BD}$, and $\overline{DA}$.

Step 3 Measure the five angles A through E at the star points. Use the calculator to find the sum of the angle measures.

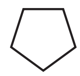
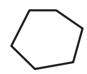
Step 4 Drag each vertex of the star and observe what happens to the angle measures and the calculated sum. Does the sum change? What is the sum?

Step 5 Copy the table on page 265. Use the Polygon Sum Conjecture to complete the first column. Then enter the angle sum for the 5-pointed star.

Step 6 Repeat Steps 1–5 for a 6-pointed star. Enter the angle sum in the table. Complete the column for each n-pointed star with every second point connected.

Step 7 What happens if you connect every third point to form a star? What would be the sum of the angle measures in this star? Complete the table column for every third point.

Step 8 Use what you have learned to complete the table. What patterns do you notice? Write the rules for n-pointed stars.

	Angle measure sums by how the star points are connected				
Number of star points	Every point	Every 2nd point	Every 3rd point	Every 4th point	Every 5th point
5	 540°				
6	 720°				
7					

Step 9 Let's explore Step 4 a little further. Can you drag the vertices of each star polygon to make it convex? Describe the steps for turning each one into a convex polygon, and then back into a star polygon again, in the fewest steps possible.

Step 10 In Step 9, how did the sum of the angle measure change when a polygon became convex? When did it change?

This blanket by Teresa Archuleta-Sagel is titled *My Blue Vallero Heaven.* Are these star polygons? Why?

Kite and Trapezoid Properties

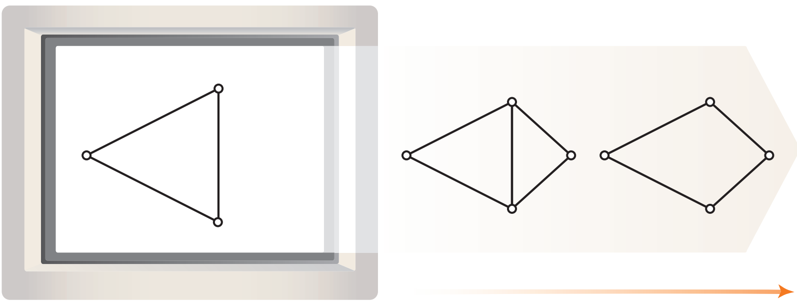
Imagination is the highest kite we fly.

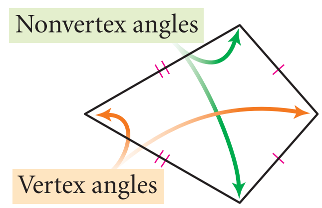
LAUREN BACALL

Recall that a **kite** is a quadrilateral with exactly two distinct pairs of congruent consecutive sides.

If you construct two different isosceles triangles on opposite sides of a common base and then remove the base, you have constructed a kite. In an isosceles triangle, the vertex angle is the angle between the two congruent sides. Therefore, let's call the two angles between each pair of congruent sides of a kite the **vertex angles** of the kite. Let's call the other pair the **nonvertex angles.**

For an interactive version of this sketch, visit www.keymath.com/DG .





A kite also has one line of reflectional symmetry, just like an isosceles triangle. You can use this property to discover other properties of kites. Let's investigate.

Investigation 1
What Are Some Properties of Kites?

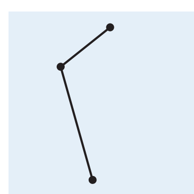
You will need

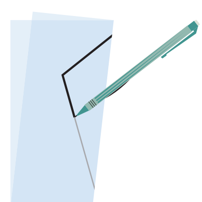
- patty paper

In this investigation you will look at angles and diagonals in a kite to see what special properties they have.

Step 1 On patty paper, draw two connected segments of different lengths, as shown. Fold through the endpoints and trace the two segments on the back of the patty paper.

Step 2 Compare the size of each pair of opposite angles in your kite by folding an angle onto the opposite angle. Are the vertex angles congruent? Are the nonvertex angles congruent? Share your observations with others near you and complete the conjecture.


Step 1

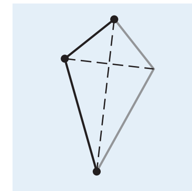

Step 2

Kite Angles Conjecture

C-35

The _?_ angles of a kite are _?_.

Step 3 Draw the diagonals. How are the diagonals related? Share your observations with others in your group and complete the conjecture.



Kite Diagonals Conjecture

C-36

The diagonals of a kite are _?_.

What else seems to be true about the diagonals of kites?

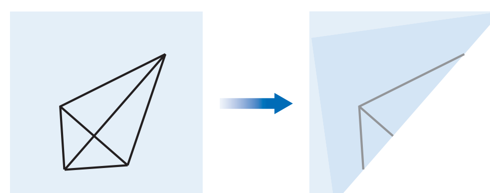
Step 4 Compare the lengths of the segments on both diagonals. Does either diagonal bisect the other? Share your observations with others near you. Copy and complete the conjecture.

Kite Diagonal Bisector Conjecture

C-37

The diagonal connecting the vertex angles of a kite is the _?_ of the other diagonal.

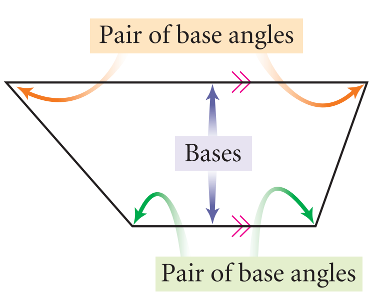
Step 5 Fold along both diagonals. Does either diagonal bisect any angles? Share your observations with others and complete the conjecture.



Kite Angle Bisector Conjecture

C-38

The _?_ angles of a kite are _?_ by a _?_.



You will prove the Kite Diagonal Bisector Conjecture and the Kite Angle Bisector Conjecture as exercises after this lesson.

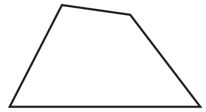
Let's move on to trapezoids. Recall that a **trapezoid** is a quadrilateral with exactly one pair of parallel sides.

In a trapezoid the parallel sides are called **bases.** A pair of angles that share a base as a common side are called **base angles.**

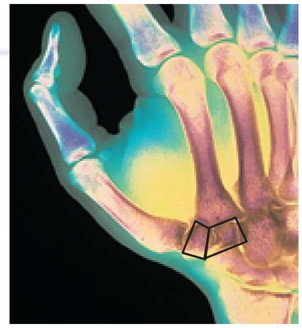
In the next investigation, you will discover some properties of trapezoids.

Science CONNECTION

A *trapezium* is a quadrilateral with *no* two sides parallel. The words *trapezoid* and *trapezium* come from the Greek word *trapeza*, meaning table. There are bones in your wrists that anatomists call trapezoid and trapezium because of their geometric shapes.



Trapezium



Investigation 2
What Are Some Properties of Trapezoids?

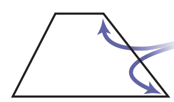
You will need

- a straightedge
- a protractor
- a compass

This is a view inside a deflating hot-air balloon. Notice the trapezoidal panels that make up the balloon.

Step 1 Use the two edges of your straightedge to draw parallel segments of unequal length. Draw two nonparallel sides connecting them to make a trapezoid.

Step 2 Use your protractor to find the sum of the measures of each pair of consecutive angles between the parallel bases. What do you notice about this sum? Share your observations with your group.

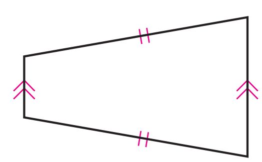


Step 3 Copy and complete the conjecture.

Trapezoid Consecutive Angles Conjecture **C-39**

The consecutive angles between the bases of a trapezoid are __?__.

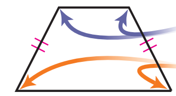
Recall from Chapter 3 that a trapezoid whose two nonparallel sides are the same length is called an **isosceles trapezoid.** Next, you will discover a few properties of isosceles trapezoids.



Like kites, isosceles trapezoids have one line of reflectional symmetry. Through what points does the line of symmetry pass?

Step 4 Use both edges of your straightedge to draw parallel lines. Using your compass, construct two congruent segments. Connect the four segments to make an isosceles trapezoid.

Step 5 Measure each pair of base angles. What do you notice about the pair of base angles in each trapezoid? Compare your observations with others near you.



Step 6 Copy and complete the conjecture.

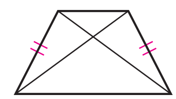
Isosceles Trapezoid Conjecture

C-40

The base angles of an isosceles trapezoid are ___?___.

What other parts of an isosceles trapezoid are congruent? Let's continue.

Step 7 Draw both diagonals. Compare their lengths. Share your observations with others near you.



Step 8 Copy and complete the conjecture.

Isosceles Trapezoid Diagonals Conjecture

C-41

The diagonals of an isosceles trapezoid are ___?___.

Suppose you assume that the Isosceles Trapezoid Conjecture is true. What pair of triangles and which triangle congruence conjecture would you use to explain why the Isosceles Trapezoid Diagonals Conjecture is true?

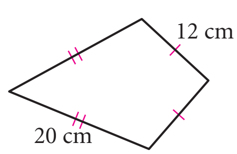
EXERCISES

You will need

Construction tools for Exercises **10–12**

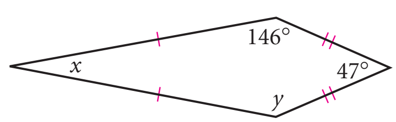
Use your new conjectures to find the missing measures.

1. Perimeter = ___?___



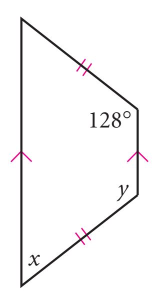
2. x = ___?___

y = ___?___



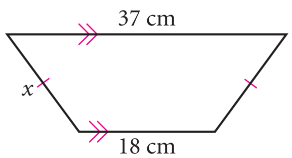
3. x = ___?___

y = ___?___



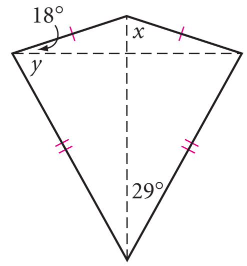
4. $x = \underline{\ ?\ }$

Perimeter $= 85$ cm



5. $x = \underline{\ ?\ }$

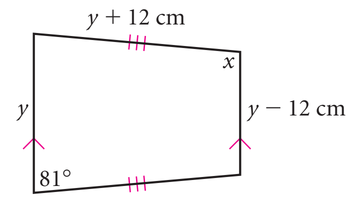
$y = \underline{\ ?\ }$



6. $x = \underline{\ ?\ }$

$y = \underline{\ ?\ }$

Perimeter $= 164$ cm



7. Sketch and label kite *KITE* with vertex angles $\angle K$ and $\angle T$ and $KI > TE$. Which angles are congruent?

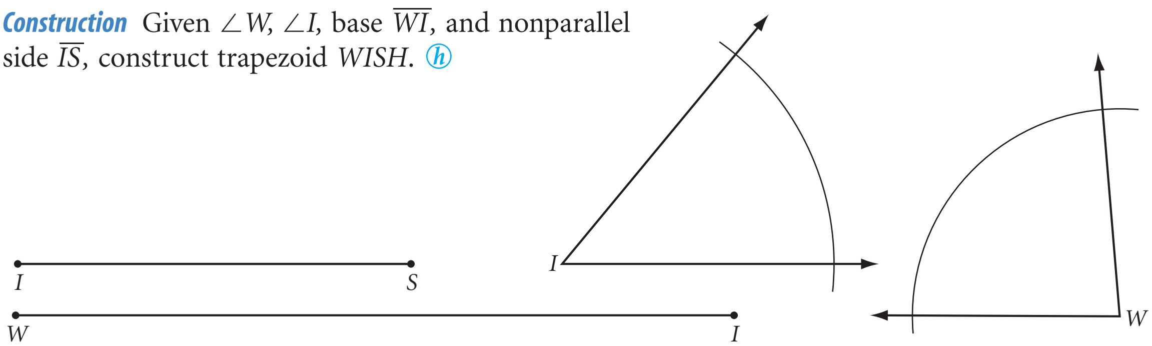
8. Sketch and label trapezoid *QUIZ* with one base $\overline{QU}$. What is the other base? Name the two pairs of base angles.

9. Sketch and label isosceles trapezoid *SHOW* with one base $\overline{SH}$. What is the other base? Name the two pairs of base angles. Name the two sides of equal length.

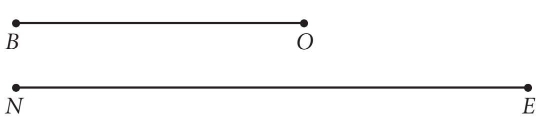
In Exercises 10–12, use the properties of kites and trapezoids to construct each figure. You may use either patty paper or a compass and a straightedge.

10. ***Construction*** Construct kite *BENF* given sides $\overline{BE}$ and $\overline{EN}$ and diagonal $\overline{BN}$. How many different kites are possible?

11. ***Construction*** Given $\angle W$, $\angle I$, base $\overline{WI}$, and nonparallel side $\overline{IS}$, construct trapezoid *WISH*. ⓗ



12. ***Construction*** Construct a trapezoid *BONE* with $\overline{BO} \parallel \overline{NE}$. How many different trapezoids can you construct?

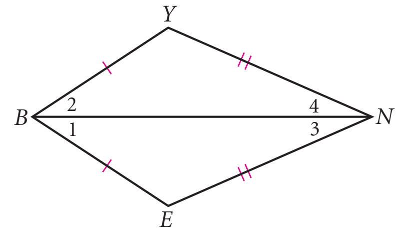


13. Write a paragraph proof or flowchart proof showing how the Kite Diagonal Bisector Conjecture logically follows from the Converse of the Perpendicular Bisector Conjecture. ⓗ

14. Copy and complete the flowchart to show how the Kite Angle Bisector Conjecture follows logically from one of the triangle congruence conjectures.

Given: Kite $BENY$ with $\overline{BE} \cong \overline{BY}$, $\overline{EN} \cong \overline{YN}$

Show: $\overline{BN}$ bisects $\angle B$
$\overline{BN}$ bisects $\angle N$

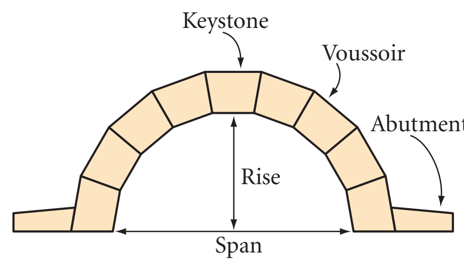


Flowchart Proof

1. $\overline{BE} \cong \overline{BY}$ — Given
2. $\overline{EN} \cong \overline{YN}$ — Given
3. $\underline{?} \cong \underline{?}$ — Same segment
4. $\triangle BEN \cong \underline{?}$ — $\underline{?}$ Congruence shortcut
5. $\angle 1 \cong \underline{?}$ and $\angle 3 \cong \underline{?}$ — $\underline{?}$
6. $\underline{?}$ and $\underline{?}$ — Definition of angle bisector

Architecture CONNECTION

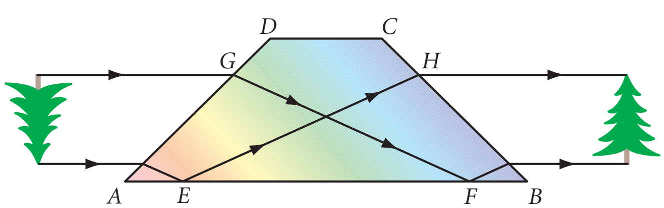
The Romans used the classical arch design in bridges, aqueducts, and buildings in the early centuries of the Common Era. The classical semicircular arch is really half of a regular polygon built with wedge-shaped blocks whose faces are isosceles trapezoids. Each block supports the blocks surrounding it.



15. **APPLICATION** The inner edge of the arch in the diagram above right is half of a regular 18-gon. Calculate the measures of all the angles in the nine isosceles trapezoids making up the arch. Then use your geometry tools to accurately draw a nine-stone arch like the one shown.

This carton is shaped like an isosceles trapezoid block.

16. The figure below shows the path of light through a trapezoidal prism, and how an image is inverted. For the prism to work as shown, the trapezoid must be isosceles, $\angle AGF$ must be congruent to $\angle BHE$, and $\overline{GF}$ must be congruent to $\overline{EH}$. Show that if these conditions are met, then $\overline{AG}$ will be congruent to $\overline{BH}$. ⓗ

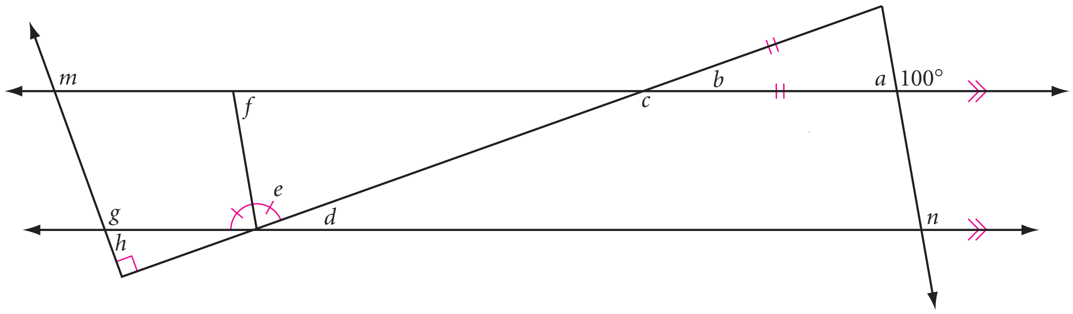


Science CONNECTION

The magnifying lenses of binoculars invert the objects you view through them, so trapezoidal prisms are used to flip the inverted images right-side-up again.

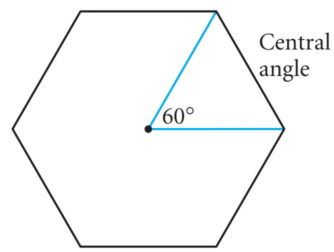
Review

17. Trace the figure below. Calculate the measure of each lettered angle.



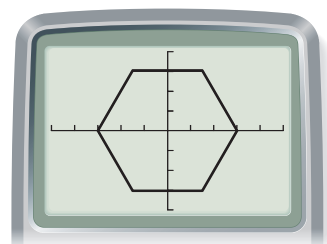
project

DRAWING REGULAR POLYGONS



You can draw a regular polygon's central angle by extending segments from the center of the polygon to its consecutive vertices. For example, the measure of each central angle of a hexagon is 60°.

Using central angles, you can draw regular polygons on a graphing calculator. This is done with parametric equations, which give the x- and y-coordinates of a point in terms of a third variable, or parameter, t.



Set your calculator's mode to degrees and parametric. Set a friendly window with an x-range of -4.7 to 4.7 and a y-range of -3.1 to 3.1. Set a t-range of 0 to 360, and t-step of 60. Enter the equations $x = 3\cos t$ and $y = 3\sin t$, and graph them. You should get a hexagon.

The equations you graphed are actually the parametric equations for a circle. By using a t-step of 60 for t-values from 0 to 360, you tell the calculator to compute only six points for the circle.

Use your calculator to investigate the following. Summarize your findings.

- Choose different t-steps to draw different regular polygons, such as an equilateral triangle, a square, a regular pentagon, and so on. What is the measure of each central angle of an n-gon?
- What happens as the measure of each central angle of a regular polygon decreases?
- What happens as you draw polygons with more and more sides?
- Experiment with rotating your polygons by choosing different t-min and t-max values. For example, set a t-range of -45 to 315, then draw a square.
- Find a way to draw star polygons on your calculator. Can you explain how this works?

LESSON

5.4

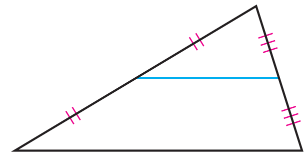
Properties of Midsegments

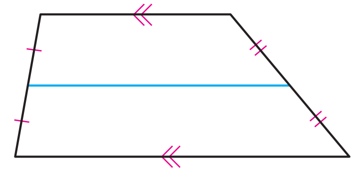
Research is formalized curiosity. It is poking and prying with a purpose.

ZORA NEALE HURSTON

As you learned in Chapter 3, the segment connecting the midpoints of two sides of a triangle is the midsegment of a triangle. The segment connecting the midpoints of the two nonparallel sides of a trapezoid is also called the midsegment of a trapezoid.

In this lesson you will discover special properties of midsegments.





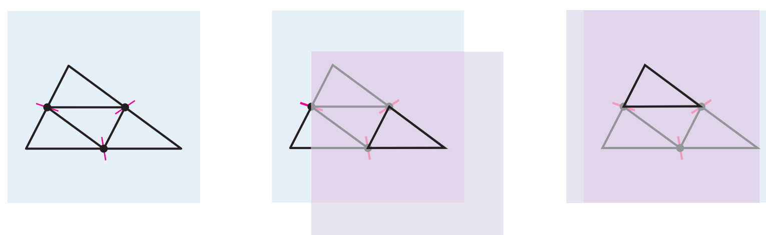
Investigation 1

Triangle Midsegment Properties

You will need

- patty paper

In this investigation you will discover two properties of the midsegment of a triangle. Each person in your group can investigate a different triangle.



Step 1 Step 2 Step 3

Step 1 Draw a triangle on a piece of patty paper. Pinch the patty paper to locate midpoints of the sides. Draw the midsegments. You should now have four small triangles.

Step 2 Place a second piece of patty paper over the first and copy one of the four triangles.

Step 3 Compare all four triangles by sliding the copy of one small triangle over the other three triangles. Compare your results with the results of your group. Copy and complete the conjecture.

Three Midsegments Conjecture C-42

The three midsegments of a triangle divide it into $\underline{\ ?\ }$.

Step 4 Mark all the congruent angles in your drawing. What conclusions can you make about each midsegment and the large triangle's third side, using the Corresponding Angles Conjecture and the Alternate Interior Angles Conjecture? What do the other students in your group think?

Step 5 Compare the length of the midsegment to the large triangle's third side. How do they relate? Copy and complete the conjecture.

Triangle Midsegment Conjecture C-43

A midsegment of a triangle is _?_ to the third side and _?_ the length of _?_.

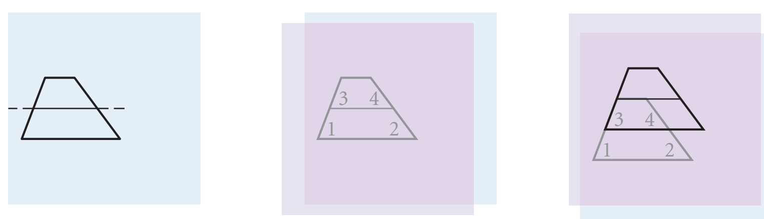
In the next investigation, you will discover two properties of the midsegment of a trapezoid.

Investigation 2
Trapezoid Midsegment Properties

You will need

- patty paper

Each person in your group can investigate a different trapezoid. Make sure you draw the two bases perfectly parallel.



Step 1 Step 2 Step 3

Step 1 Draw a small trapezoid on the left side of a piece of patty paper. Pinch the paper to locate the midpoints of the nonparallel sides. Draw the midsegment.

Step 2 Label the angles as shown. Place a second piece of patty paper over the first and copy the trapezoid and its midsegment.

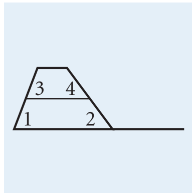
Step 3 Compare the trapezoid's base angles with the corresponding angles at the midsegment by sliding the copy up over the original.

Step 4 Are the corresponding angles congruent? What can you conclude about the midsegment and the bases? Compare your results with the results of other students.

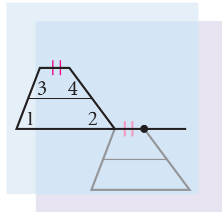
The midsegment of a triangle is half the length of the third side. How does the length of the midsegment of a trapezoid compare to the lengths of the two bases? Let's investigate.

Step 5 On the original trapezoid, extend the longer base to the right by at least the length of the shorter base.

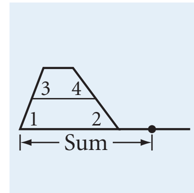
Step 6 Slide the second patty paper under the first. Show the sum of the lengths of the two bases by marking a point on the extension of the longer base.



Step 5



Step 6



Step 7

Step 7 How many times does the midsegment fit onto the segment representing the sum of the lengths of the two bases? What do you notice about the length of the midsegment and the sum of the lengths of the two bases?

Step 8 Combine your conclusions from Steps 4 and 7 and complete this conjecture.

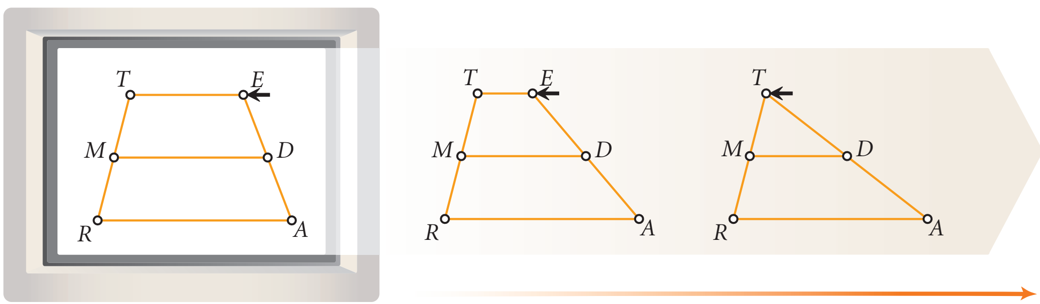
Trapezoid Midsegment Conjecture

C-44

The midsegment of a trapezoid is _?_ to the bases and is equal in length to _?_.

What happens if one base of the trapezoid shrinks to a point? Then the trapezoid collapses into a triangle, the midsegment of the trapezoid becomes a midsegment of the triangle, and the Trapezoid Midsegment Conjecture becomes the Triangle Midsegment Conjecture. Do both of your midsegment conjectures work for the last figure?

For an interactive version of this sketch, visit www.keymath.com/DG .



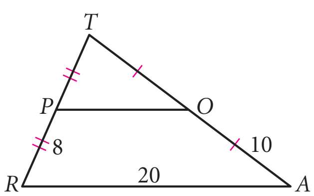
Exercises

You will need

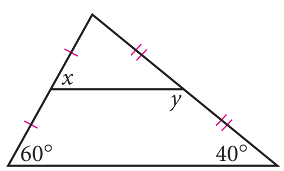
Construction tools for Exercises 9 and 18

1. How many midsegments does a triangle have? A trapezoid have?

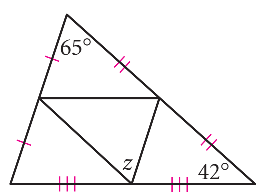
2. What is the perimeter of $\triangle TOP$? (h)



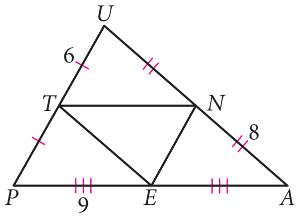
3. $x =$ _?_
$y =$ _?_



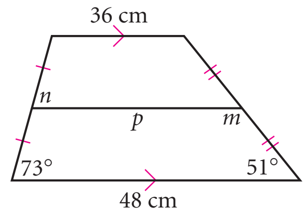
4. $z =$ _?_ (h)



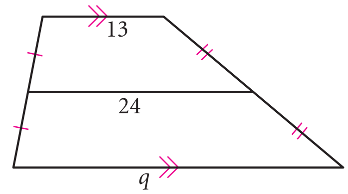
5. What is the perimeter of $\triangle TEN$?



6. $m = \underline{?}$
$n = \underline{?}$
$p = \underline{?}$



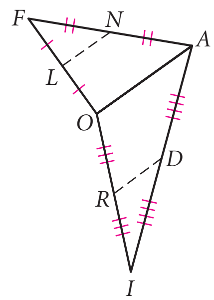
7. $q = \underline{?}$



8. Copy and complete the flowchart to show that $\overline{LN} \parallel \overline{RD}$.

Given: Midsegment $\overline{LN}$ in $\triangle FOA$
Midsegment $\overline{RD}$ in $\triangle IOA$

Show: $\overline{LN} \parallel \overline{RD}$



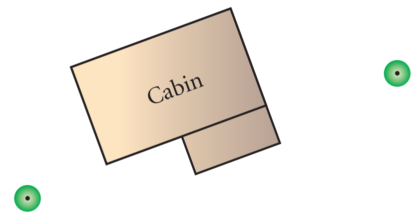
Flowchart Proof

1. $\triangle FOA$ with midsegment $\overline{LN}$ — Given
2. $\triangle IOA$ with midsegment $\overline{RD}$ — Given
3. $\overline{LN} \parallel \overline{OA}$ — $\underline{?}$
4. $\underline{?}$ — Triangle Midsegment Conjecture
5. $\underline{?}$ — Two lines parallel to the same line are parallel

9. *Construction* When you connected the midpoints of the three sides of a triangle in Investigation 1, you created four congruent triangles. Draw a quadrilateral on patty paper and pinch the paper to locate the midpoints of the four sides. Connect the midpoints to form a quadrilateral. What special type of quadrilateral do you get when you connect the midpoints? Use the Triangle Midsegment Conjecture to explain your answer. ⓗ

10. Deep in a tropical rain forest, archaeologist Ertha Diggs and her assistant researchers have uncovered a square-based truncated pyramid (a square pyramid with the top part removed). The four lateral faces are isosceles trapezoids. A line of darker mortar runs along the midsegment of each lateral face. Ertha and her co-workers make some measurements and find that one of these midsegments measures 41 meters and each bottom base measures 52 meters. Now that they have this information, Ertha and her team can calculate the length of the top base without having to climb up and measure it. Can you? What is the length of the top edge? How do you know?

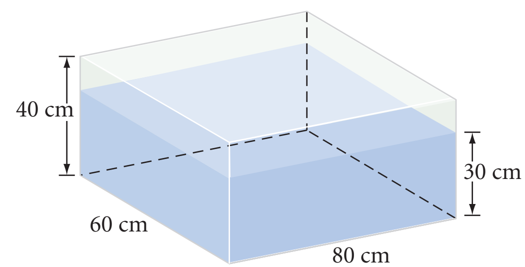
11. Ladie and Casey pride themselves on their estimation skills and take turns estimating distances. Casey claims that two large redwood trees visible from where they are sitting are 180 feet apart, and Ladie says they are 275 feet apart.



The problem is, they can't measure the distance to see whose estimate is better, because their cabin is located between the trees. All of a sudden, Ladie recalls her geometry: "Oh yeah, the Triangle Midsegment Conjecture!" She collects a tape measure, a hammer, and some wooden stakes. What is she going to do?

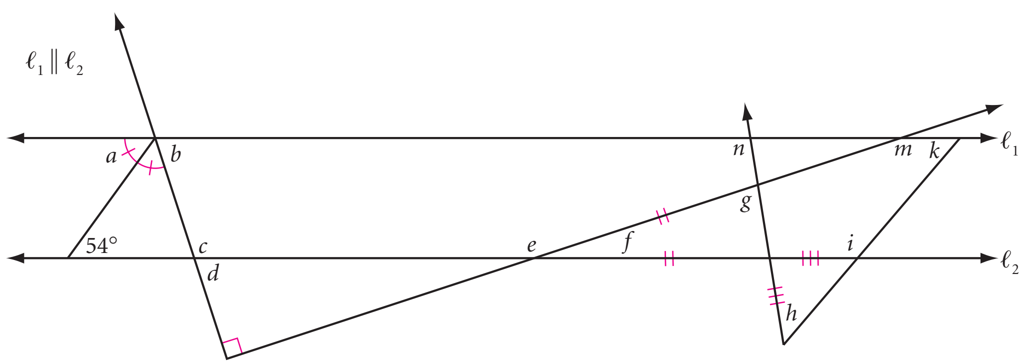
Review

12. The 40-by-60-by-80 cm sealed rectangular container shown at right is resting on its largest face. It is filled with a liquid to a height of 30 cm. Sketch the container resting on its smallest face. Show the height of the liquid in this new position.

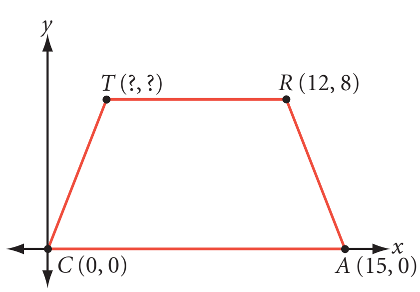


13. Write the converse of this statement: If exactly one diagonal bisects a pair of opposite angles of a quadrilateral, then the quadrilateral is a kite. Is the converse true? Is the original statement true? If either conjecture is not true, sketch a counterexample.

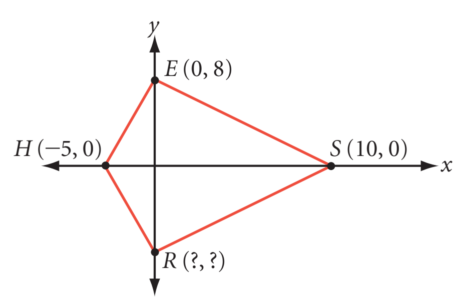
14. Trace the figure below. Calculate the measure of each lettered angle.



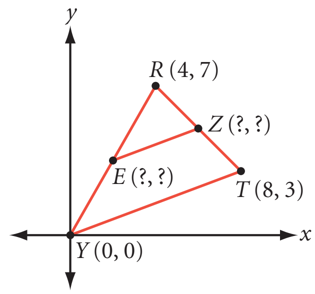
15. *CART* is an isosceles trapezoid. What are the coordinates of point *T*?



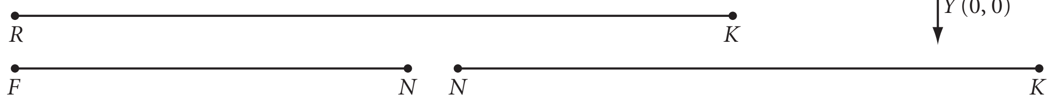
16. *HRSE* is a kite. What are the coordinates of point *R*?



17. Find the coordinates of midpoints E and Z. Show that the slope of the line containing midsegment $\overline{EZ}$ is equal to the slope of the line containing $\overline{YT}$.



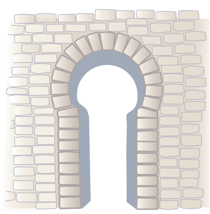
18. ***Construction*** Use the kite properties you discovered in Lesson 5.3 to construct kite $FRNK$ given diagonals $\overline{RK}$ and $\overline{FN}$ and side $\overline{NK}$. Is there only one solution?



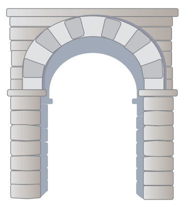
project

BUILDING AN ARCH

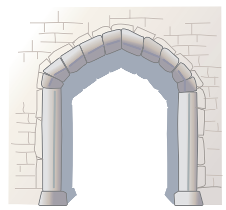
In this project, you'll design and build your own Roman arch.



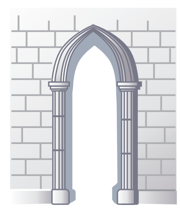
Horseshoe Arch



Basket Arch



Tudor Arch



Lancet Arch

Arches can have a simple semicircular shape, or a pointed "broken arch" shape.

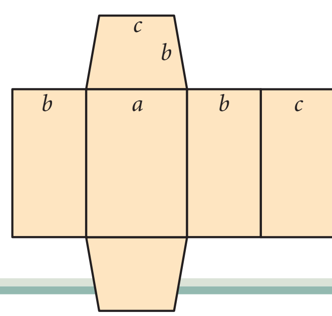
In arch construction, a wooden support holds the voussoirs in place until the keystone is placed (see arch diagram on page 271). It's said that when the Romans made an arch, they would make the architect stand under it while the wooden support was removed. That was one way to be sure architects carefully designed arches that wouldn't fall!

The arches in this Roman aqueduct, above the Gard River in France, are typical of arches you can find throughout regions that were once part of the Roman Empire. An arch can carry a lot of weight, yet it also provides an opening. The abutments on the sides of the arch keep the arch from spreading out and falling down.

What size arch would you like to build? Decide the dimensions of the opening, the thickness of the arch, and the number of voussoirs. Decide on the materials you will use. You should have your trapezoid and your materials approved by your group or your teacher before you begin construction.

Your project should include

- A scale diagram that shows the exact size and angle of the voussoirs and the keystone.
- A template for your voussoirs.
- Your arch.



LESSON 5.5

Properties of Parallelograms

If there is an opinion, facts will be found to support it.

JUDY SPROLES

In this lesson you will discover some special properties of parallelograms. A parallelogram is a quadrilateral whose opposite sides are parallel.

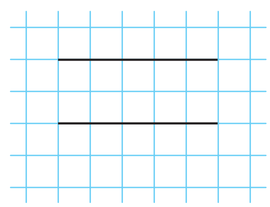
Rhombuses, rectangles, and squares all fit this definition as well. Therefore, any properties you discover for parallelograms will also apply to these other shapes. However, to be sure that your conjectures will apply to *any* parallelogram, you should investigate parallelograms that don't have any other special properties, such as right angles, all congruent angles, or all congruent sides.

Investigation
Four Parallelogram Properties

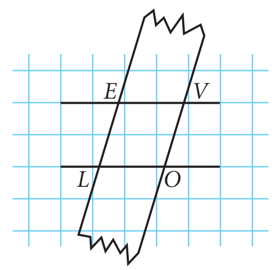
You will need

- graph paper
- patty paper or a compass
- a straightedge
- a protractor

First you'll create a parallelogram.



Step 1



Step 2

Step 1 Using the lines on a piece of graph paper as a guide, draw a pair of parallel lines that are at least 6 cm apart. Using the parallel edges of your straightedge, make a parallelogram. Label your parallelogram *LOVE*.

Step 2 Let's look at the opposite angles. Measure the angles of parallelogram *LOVE*. Compare a pair of opposite angles using patty paper or your protractor.

Compare results with your group. Copy and complete the conjecture.

Parallelogram Opposite Angles Conjecture C-45

The opposite angles of a parallelogram are ?.

Two angles that share a common side in a polygon are consecutive angles. In parallelogram *LOVE*, $\angle LOV$ and $\angle EVO$ are a pair of consecutive angles. The consecutive angles of a parallelogram are also related.

Step 3 Find the sum of the measures of each pair of consecutive angles in parallelogram *LOVE*.

Share your observations with your group. Copy and complete the conjecture.

Parallelogram Consecutive Angles Conjecture C-46

The consecutive angles of a parallelogram are _?_.

Step 4 Describe how to use the two conjectures you just made to find all the angles of a parallelogram with only one angle measure given.

Step 5 Next let's look at the opposite sides of a parallelogram. With your compass or patty paper, compare the lengths of the opposite sides of the parallelogram you made.

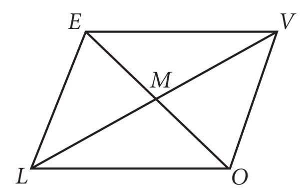
Share your results with your group. Copy and complete the conjecture.

Parallelogram Opposite Sides Conjecture C-47

The opposite sides of a parallelogram are _?_.

Step 6 Finally, let's consider the diagonals of a parallelogram. Construct the diagonals $\overline{LV}$ and $\overline{EO}$, as shown below. Label the point where the two diagonals intersect point M.

Step 7 Measure LM and VM. What can you conclude about point M? Is this conclusion also true for diagonal $\overline{EO}$? How do the diagonals relate?



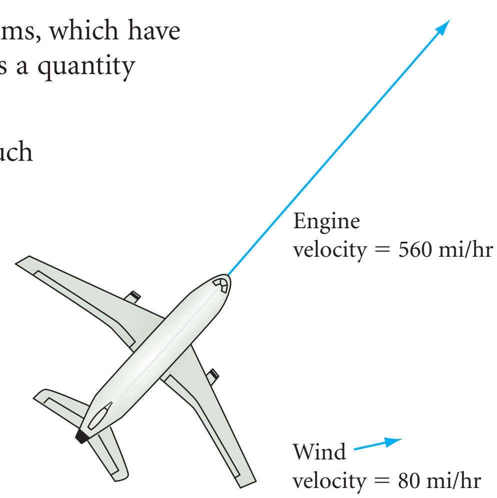
Share your results with your group. Copy and complete the conjecture.

Parallelogram Diagonals Conjecture C-48

The diagonals of a parallelogram _?_.

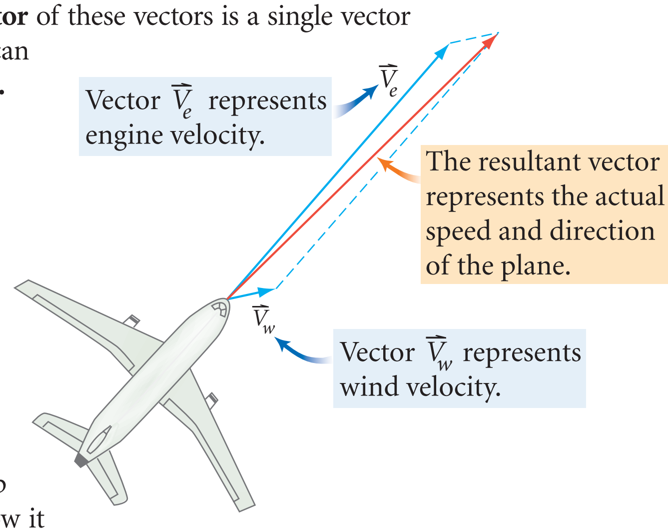
Parallelograms are used in vector diagrams, which have many applications in science. A **vector** is a quantity that has both magnitude and direction.

Vectors describe quantities in physics, such as velocity, acceleration, and force. You can represent a vector by drawing an arrow. The length and direction of the arrow represent the magnitude and direction of the vector. For example, a velocity vector tells you an airplane's speed and direction. The lengths of vectors in a diagram are proportional to the quantities they represent.



In many physics problems, you combine vector quantities acting on the same object. For example, the wind current and engine thrust vectors determine the velocity of an airplane. The **resultant vector** of these vectors is a single vector that has the same effect. It can also be called a **vector sum.** To find a resultant vector, make a parallelogram with the vectors as sides. The resultant vector is the diagonal of the parallelogram from the two vectors' tails to the opposite vertex.

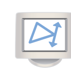
In the diagram at right, the resultant vector shows that the wind will speed up the plane, and will also blow it slightly off course.



EXERCISES

You will need

Construction tools for Exercises **7** and **8**

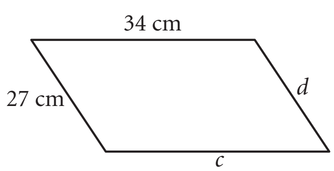


Geometry software for Exercises **21** and **22**

Use your new conjectures in the following exercises. In Exercises 1–6, each figure is a parallelogram.

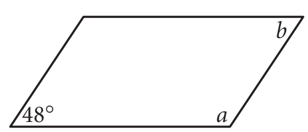
1. $c = \underline{?}$

$d = \underline{?}$



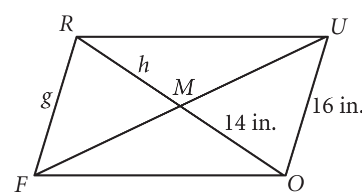
2. $a = \underline{?}$

$b = \underline{?}$



3. $g = \underline{?}$

$h = \underline{?}$

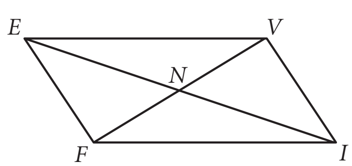


4. $VF = 36$ m

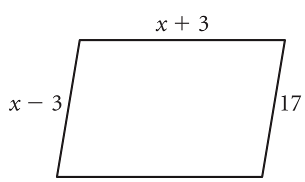
$EF = 24$ m

$EI = 42$ m

What is the perimeter of $\triangle NVI$? ⓗ

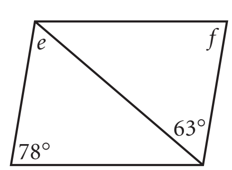


5. What is the perimeter?

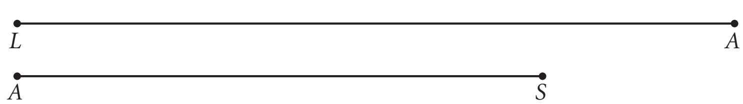


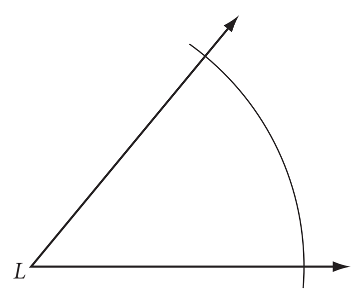
6. $e = \underline{?}$

$f = \underline{?}$

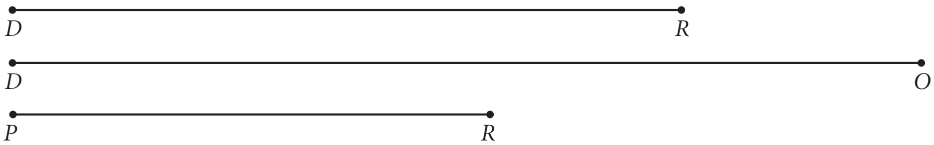


7. ***Construction*** Given side $\overline{LA}$, side $\overline{AS}$, and $\angle L$, construct parallelogram $LAST$.



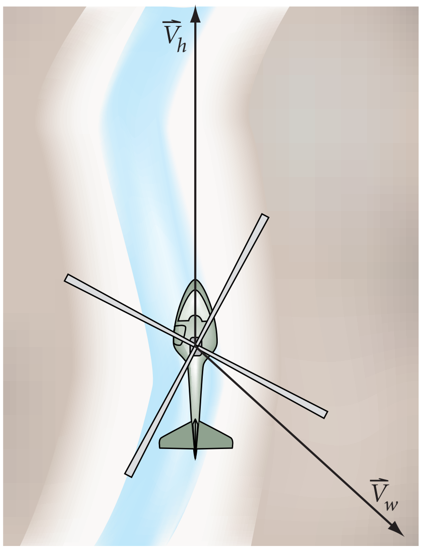


8. ***Construction*** Given side $\overline{DR}$ and diagonals $\overline{DO}$ and $\overline{PR}$, construct parallelogram *DROP*. ⓗ

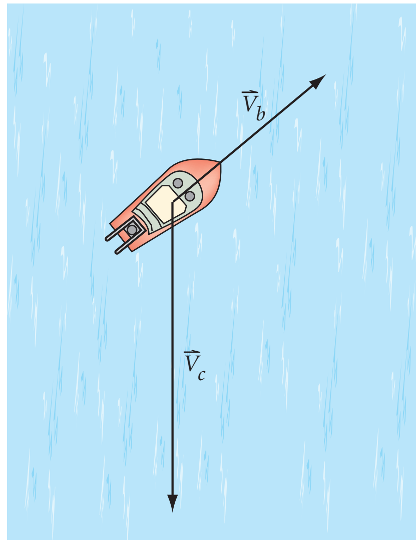


In Exercises 9 and 10, copy the vector diagram and draw the resultant vector.

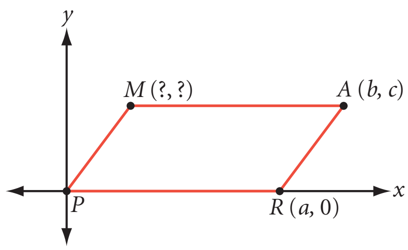
9.



10. ⓗ



11. Find the coordinates of point *M* in parallelogram *PRAM*. ⓗ

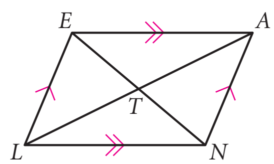


12. Draw a quadrilateral. Make a copy of it. Draw a diagonal in the first quadrilateral. Draw the *other* diagonal in the duplicate quadrilateral. Cut each quadrilateral into two triangles along the diagonals. Arrange the four triangles into a parallelogram. Make a sketch showing how you did it.

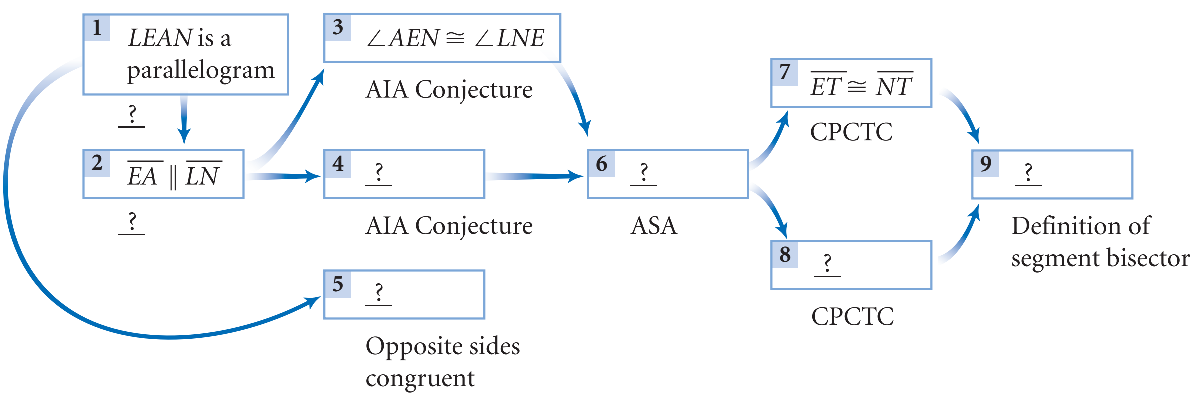
13. Copy and complete the flowchart to show how the Parallelogram Diagonals Conjecture follows logically from other conjectures.

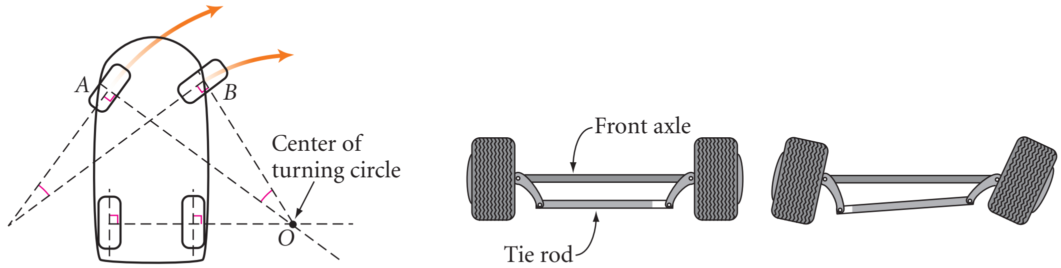
Given: *LEAN* is a parallelogram

Show: $\overline{EN}$ and $\overline{LA}$ bisect each other



Flowchart Proof





Trapezoid linkage (Top view)

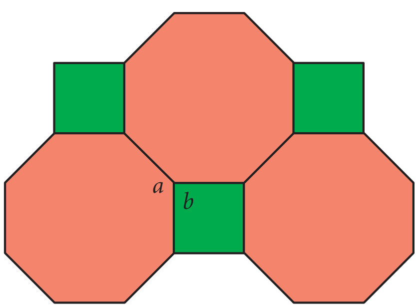
Technology CONNECTION

Quadrilateral linkages are used in mechanical design, robotics, the automotive industry, and toy making. In cars, they are used to turn each front wheel the right amount for a smooth turn.

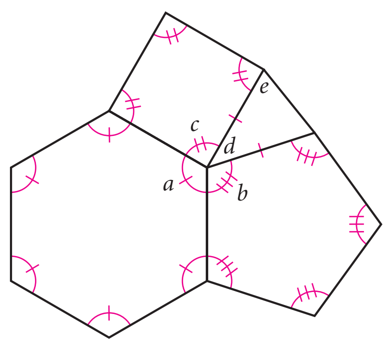
14. Study the sewing box pictured here. Sketch the box as viewed from the side, and explain why a parallelogram linkage is used.

Review

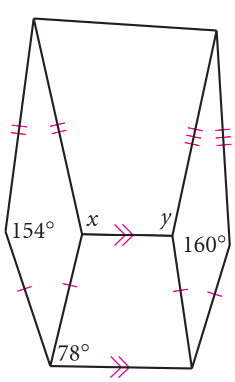
15. Find the measures of the lettered angles in this tiling of regular polygons.



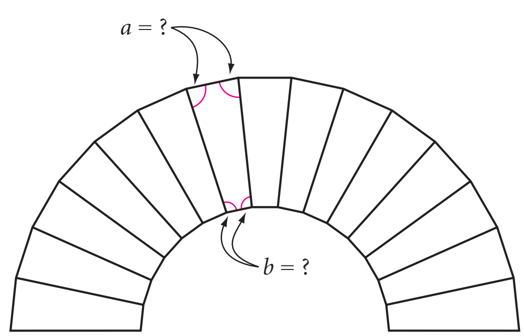
16. Trace the figure below. Calculate the measure of each lettered angle.



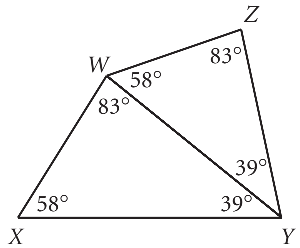
17. Find x and y. Explain.



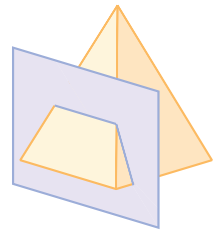
18. What is the measure of each angle in the isosceles trapezoid face of a voussoir in this 15-stone arch?



19. Is $\triangle XYW \cong \triangle WYZ$? Explain.



20. Sketch the section formed when this pyramid is sliced by the plane.



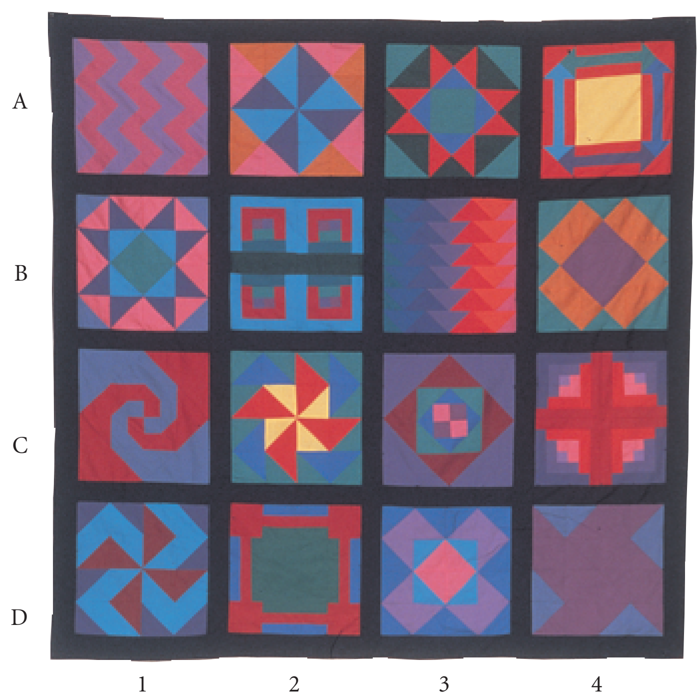
21. *Technology* Construct two segments that bisect each other. Connect their endpoints. What type of quadrilateral is this? Draw a diagram and explain why.

22. *Technology* Construct two intersecting circles. Connect the two centers and the two points of intersection to form a quadrilateral. What type of quadrilateral is this? Draw a diagram and explain why.

IMPROVING YOUR VISUAL THINKING SKILLS

A Puzzle Quilt

Fourth-grade students at Public School 95, the Bronx, New York, made the puzzle quilt at right with the help of artist Paula Nadelstern. Each square has a twin made of exactly the same shaped pieces. Only the colors, chosen from traditional Amish colors, are different. For example, square A1 is the twin of square B3. Match each square with its twin.

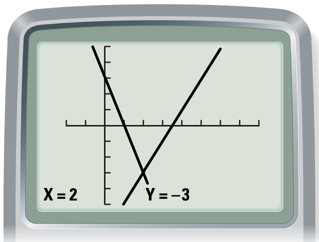


Solving Systems of Linear Equations

A *system of equations* is a set of two or more equations with the same variables. The solution of a system is the set of values that makes all the equations in the system true. For example, the system of equations below has solution (2, −3). Verify this by substituting 2 for x and −3 for y in both equations.

$$\begin{cases} y = 2x - 7 \\ y = -3x + 3 \end{cases}$$

Graphically, the solution of a system is the point of intersection of the graphs of the equations.



You can estimate the solution of a system by graphing the equations. However, the point of intersection may not have convenient integer coordinates. To find the exact solution, you can use algebra. Examples A and B review how to use the *substitution* and *elimination* methods for solving systems of equations.

EXAMPLE A

Use the substitution method to solve the system $\begin{cases} 3y = 12x - 21 \\ 12x + 2y = 1 \end{cases}$.

▸ Solution

Start by solving the first equation for y to get $y = 4x - 7$.

Now, substitute the expression $4x - 7$ from the resulting equation for y in the second original equation.

$12x + 2y = 1$ — Second original equation.

$12x + 2(4x - 7) = 1$ — Substitute $4x - 7$ for y.

$x = \frac{3}{4}$ — Solve for x.

To find y, substitute $\frac{3}{4}$ for x in either original equation.

$3y = 12\left(\frac{3}{4}\right) - 21$ — Substitute $\frac{3}{4}$ for x in the first original equation.

$y = -4$ — Solve for y.

The solution of the system is $\left(\frac{3}{4}, -4\right)$. Verify by substituting these values for x and y in each of the original equations.

EXAMPLE B

The band sold calendars to raise money for new uniforms. Aisha sold 6 desk calendars and 10 wall calendars for a total of \$100. Ted sold 12 desk calendars and 4 wall calendars for a total of \$88. Find the price of each type of calendar by writing a system of equations and solving it using the elimination method.

▸ Solution

Let d be the price of a desk calendar, and let w be the price of a wall calendar. You can write this system to represent the situation.

$$\begin{cases} 6d + 10w = 100 & \text{Aisha's sales.} \\ 12d + 4w = 88 & \text{Ted's sales.} \end{cases}$$

Solving a system by elimination involves adding or subtracting the equations to eliminate one of the variables. To solve this system, first multiply both sides of the first equation by 2.

$$\begin{cases} 6d + 10w = 100 \\ 12d + 4w = 88 \end{cases} \rightarrow \begin{cases} 12d + 20w = 200 \\ 12d + 4w = 88 \end{cases}$$

Now, subtract the second equation from the first to eliminate d.

$$\begin{array}{rl} 12d + 20w &= 200 \\ -(12d + 4w &= 88) \\ \hline 16w &= 112 \\ w &= 7 \end{array}$$

To find the value of d, substitute 7 for w in either original equation. The solution is $w = 7$ and $d = 5$, so a wall calendar costs \$7 and a desk calendar costs \$5.

EXERCISES

Solve each system of equations algebraically.

1. $\begin{cases} y = -2x + 2 \\ 6x + 2y = 3 \end{cases}$

2. $\begin{cases} x + 2y = 3 \\ 2x - y = 16 \end{cases}$

3. $\begin{cases} 5x - y = -1 \\ 15x = 2y \end{cases}$

4. $\begin{cases} -4x + 3y = 3 \\ 7x - 9y = 6 \end{cases}$

For Exercises 5 and 6 solve the systems. What happens? Graph each set of equations and use the graphs to explain your results.

5. $\begin{cases} x + 6y = 10 \\ \frac{1}{2}x + 3y = 5 \end{cases}$

6. $\begin{cases} 2x + y = 30 \\ y = -2x - 1 \end{cases}$

7. A snowboard rental company offers two different rental plans. Plan A offers \$4/hr for the rental and a \$20 lift ticket. Plan B offers \$7/hr for the rental and a free lift ticket.

a. Write the two equations that represent the costs for the two plans, using x for the number of hours. Solve for x and y.

b. Graph the two equations. What does the point of intersection represent?

c. Which is the better plan if you intend to snowboard for 5 hours? What is the most number of hours of snowboarding you can get for \$50?

8. The lines $y = 3 + \frac{2}{3}x$, $y = -\frac{1}{3}x$, and $y = -\frac{4}{3}x + 3$ intersect to form a triangle. Find the vertices of the triangle.

LESSON

5.6

Properties of Special Parallelograms

You must know a great deal about a subject to know how little is known about it.

LEO ROSTEN

The legs of the lifting platforms shown at right form rhombuses. Can you visualize how this lift would work differently if the legs formed parallelograms that weren't rhombuses?

In this lesson you will discover some properties of rhombuses, rectangles, and squares. What you discover about the diagonals of these special parallelograms will help you understand why these lifts work the way they do.

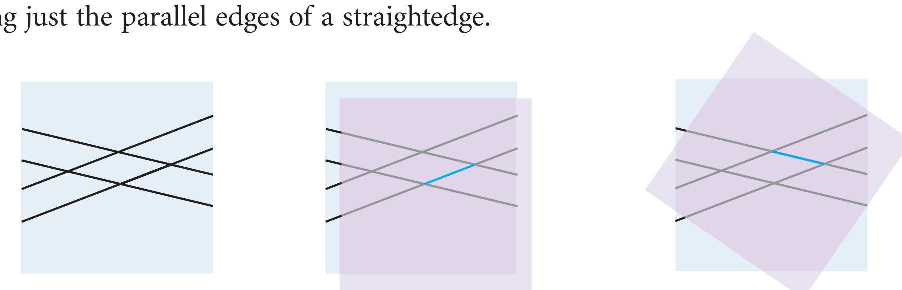
Investigation 1

What Can You Draw with the Double-Edged Straightedge?

You will need

- patty paper
- a double-edged straightedge

In this investigation you will discover the special parallelogram that you can draw using just the parallel edges of a straightedge.



Step 1 Step 2 Step 3

Step 1 On a piece of patty paper, use a double-edged straightedge to draw two pairs of parallel lines that intersect each other.

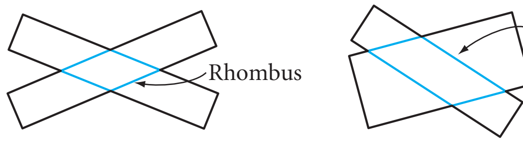
Step 2 Assuming that the two edges of your straightedge are parallel, you have drawn a parallelogram. Place a second patty paper over the first and copy one of the sides of the parallelogram.

Step 3 Compare the length of the side on the second patty paper with the lengths of the other three sides of the parallelogram. How do they compare? Share your results with your group. Copy and complete the conjecture.

Double-Edged Straightedge Conjecture C-49

If two parallel lines are intersected by a second pair of parallel lines that are the same distance apart as the first pair, then the parallelogram formed is a __?__.

In Chapter 3, you learned how to construct a rhombus using a compass and straightedge, or using patty paper. Now you know a quicker and easier way, using a double-edged straightedge. To construct a parallelogram that is *not* a rhombus, you need two double-edged staightedges of different widths.



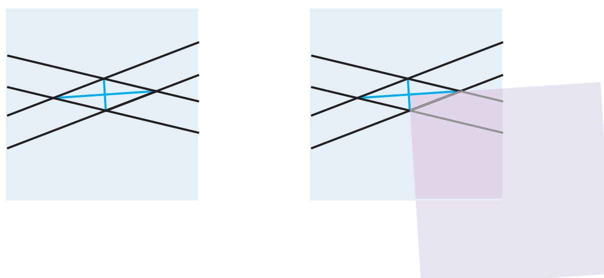
Now let's investigate some properties of rhombuses.

Investigation 2

Do Rhombus Diagonals Have Special Properties?

You will need

- patty paper



Step 1

Step 2

Step 1 Draw in both diagonals of the rhombus you created in Investigation 1.

Step 2 Place the corner of a second patty paper onto one of the angles formed by the intersection of the two diagonals. Are the diagonals perpendicular?

Compare your results with your group. Also, recall that a rhombus is a parallelogram and that the diagonals of a parallelogram bisect each other. Combine these two ideas into your next conjecture.

Rhombus Diagonals Conjecture

C-50

The diagonals of a rhombus are ___?___, and they ___?___.

Step 3 The diagonals and the sides of the rhombus form two angles at each vertex. Fold your patty paper to compare each pair of angles. What do you observe? Compare your results with your group. Copy and complete the conjecture.

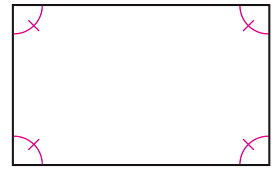
Rhombus Angles Conjecture

C-51

The ___?___ of a rhombus ___?___ the angles of the rhombus.

So far you've made conjectures about a quadrilateral with four congruent sides. Now let's look at quadrilaterals with four congruent angles. What special properties do they have?

Recall the definition you created for a rectangle.
A **rectangle** is an equiangular parallelogram.



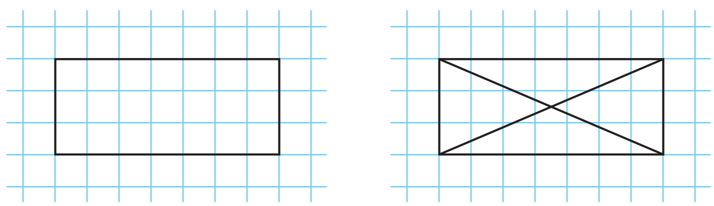
Here is a thought experiment. What is the measure of each angle of a rectangle? The Quadrilateral Sum Conjecture says all four angles add up to 360°. They're congruent, so each angle must be 90°, or a right angle.

Investigation 3
Do Rectangle Diagonals Have Special Properties?

You will need

- graph paper
- a compass

Now let's look at the diagonals of rectangles.



Step 1 Step 2

Step 1 Draw a large rectangle using the lines on a piece of graph paper as a guide.

Step 2 Draw in both diagonals. With your compass, compare the lengths of the two diagonals.

Compare results with your group. In addition, recall that a rectangle is also a parallelogram. So its diagonals also have the properties of a parallelogram's diagonals. Combine these ideas to complete the conjecture.

Rectangle Diagonals Conjecture C-52

The diagonals of a rectangle are ___?___ and ___?___.

Career CONNECTION

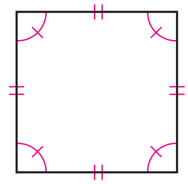
A tailor uses a button spacer to mark the locations of the buttons. The tool opens and closes, but the tips always remain equally spaced. What quadrilateral properties make this tool work correctly?

What happens if you combine the properties of a rectangle and a rhombus? We call the shape a square, and you can think of it as a regular quadrilateral. So you can define it in two different ways.

A **square** is an equiangular rhombus.

Or

A **square** is an equilateral rectangle.



A square is a parallelogram, as well as both a rectangle and a rhombus. Use what you know about the properties of these three quadrilaterals to copy and complete this conjecture.

Square Diagonals Conjecture

C-53

The diagonals of a square are _?_, _?_, and _?_.

EXERCISES

You will need

Construction tools for Exercises **17–19, 23, 24,** and **30**

For Exercises 1–10 identify each statement as true or false. For each false statement, sketch a counterexample or explain why it is false.

1. The diagonals of a parallelogram are congruent. ⓗ

2. The consecutive angles of a rectangle are congruent and supplementary.

3. The diagonals of a rectangle bisect each other.

4. The diagonals of a rectangle bisect the angles.

5. The diagonals of a square are perpendicular bisectors of each other.

6. Every rhombus is a square.

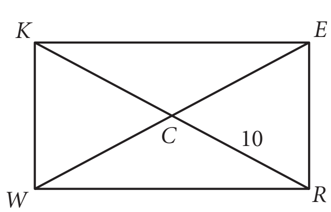
7. Every square is a rectangle.

8. A diagonal divides a square into two isosceles right triangles.

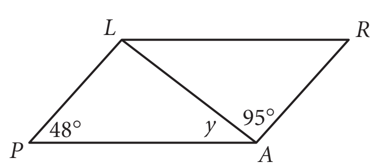
9. Opposite angles in a parallelogram are always congruent.

10. Consecutive angles in a parallelogram are always congruent.

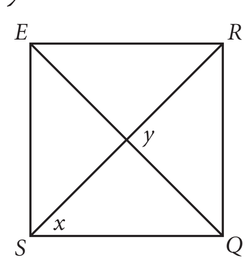
11. *WREK* is a rectangle.
$CR = 10$
$WE =$ _?_



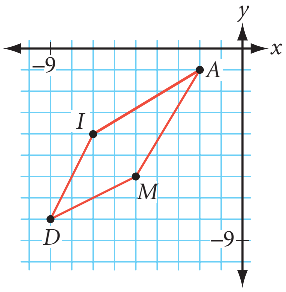
12. *PARL* is a parallelogram.
$y =$ _?_ ⓗ



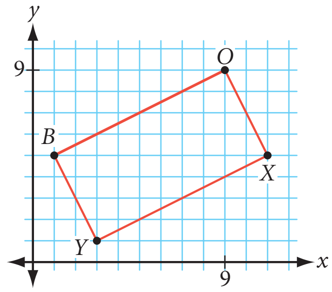
13. *SQRE* is a square.
$x =$ _?_
$y =$ _?_



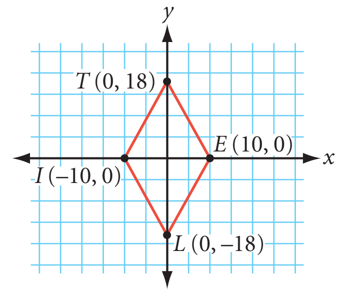
14. Is *DIAM* a rhombus? Why?



15. Is *BOXY* a rectangle? Why?

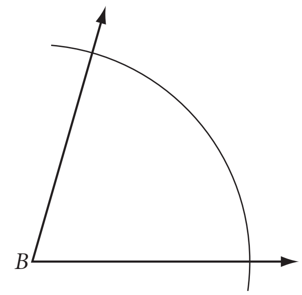


16. Is *TILE* a parallelogram? Why?

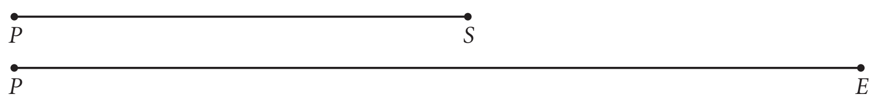


17. ***Construction*** Given the diagonal $\overline{LV}$, construct square *LOVE*. ⓗ

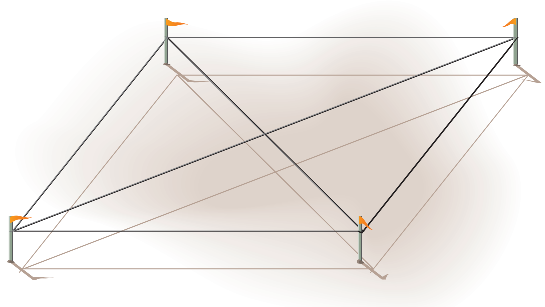
18. ***Construction*** Given diagonal $\overline{BK}$ and $\angle B$, construct rhombus *BAKE*. ⓗ



19. ***Construction*** Given side $\overline{PS}$ and diagonal $\overline{PE}$, construct rectangle *PIES*.



20. To make sure that a room is rectangular, builders check the two diagonals of the room. Explain what they must check, and why this works.



21. The platforms shown at the beginning of this lesson lift objects straight up. The platform also stays parallel to the floor. You can clearly see rhombuses in the picture, but you can also visualize the frame as the diagonals of three rectangles. Explain why the diagonals of a rectangle guarantee this vertical movement.

22. At the street intersection shown at right, one of the streets is wider than the other. Do the crosswalks form a rhombus or a parallelogram? Explain. What would have to be true about the streets if the crosswalks formed a rectangle? A square?

In Exercises 23 and 24, use only the two parallel edges of your double-edged straightedge. You may not fold the paper or use any marks on the straightedge.

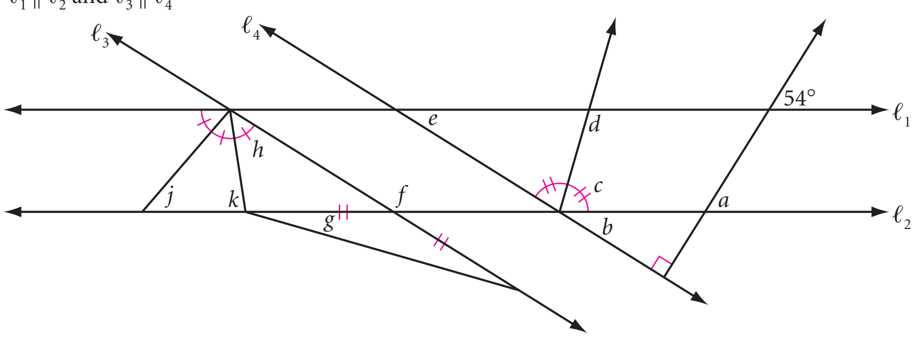
23. *Construction* Draw an angle on your paper. Use your double-edged straightedge to construct the bisector of the angle. ⓗ

24. *Construction* Draw a segment on your paper. Use your double-edged straightedge to construct the perpendicular bisector of the segment. ⓗ

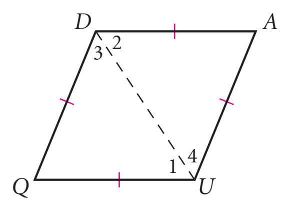
Review

25. Trace the figure below. Calculate the measure of each lettered angle.

$\ell_1 \parallel \ell_2$ and $\ell_3 \parallel \ell_4$



26. Complete the flowchart proof below to demonstrate logically that if a quadrilateral has four congruent sides then it is a rhombus. One possible proof for this argument has been started for you.



Given: Quadrilateral $QUAD$ has $\overline{QU} \cong \overline{UA} \cong \overline{AD} \cong \overline{DQ}$ with diagonal $\overline{DU}$

Show: $QUAD$ is a rhombus

Flowchart Proof

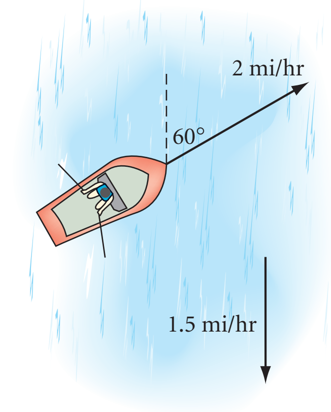
1. $\overline{QU} \cong \overline{UA}$ and $\overline{AD} \cong \overline{DQ}$ — Given
2. $\overline{QU} \cong \overline{DA}$, $\overline{QD} \cong \overline{UA}$ — ?
3. $\overline{DU} \cong \overline{DU}$ — Same segment
4. $\triangle QUD \cong \triangle ADU$ — ?
5. $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$ — ?
6. $\overline{QU} \parallel \overline{AD}$, $\overline{QD} \parallel \overline{UA}$ — If alternate interior angles are congruent, then lines are parallel
7. $QUAD$ is a parallelogram — Definition of parallelogram
8. $QUAD$ is a rhombus — ?

27. Find the coordinates of three more points that lie on the line passing through the points $(2, -1)$ and $(-3, 4)$.

28. Find the coordinates of the circumcenter and the orthocenter for $\triangle RGT$ with vertices $R(2, -1)$, $G(5, 2)$, and $T(-3, 4)$.

29. Draw a counterexample to show that this statement is false: If a triangle is isosceles, then its base angles are not complementary.

30. ***Construction*** Oran Boatwright is rowing at a 60° angle from the upstream direction as shown. Use a ruler and a protractor to draw the vector diagram. Draw the resultant vector and measure it to find his actual velocity and direction.



31. In Exercise 26, you proved that if the four sides of a quadrilateral are congruent, then the quadrilateral is a rhombus. So, when we defined rhombus, we did not need the added condition of it being a parallelogram. We only needed to say that it is a quadrilateral with all four sides congruent. Is this true for rectangles? Your conjecture would be, "If a quadrilateral has all four angles congruent, it must be a rectangle." Can you find a counterexample that proves it false? If you cannot, try to create a proof showing that it is true. ⓗ

IMPROVING YOUR REASONING SKILLS

How Did the Farmer Get to the Other Side?

A farmer was taking her pet rabbit, a basket of prize-winning baby carrots, and her small—but hungry—rabbit-chasing dog to town. She came to a river and realized she had a problem. The little boat she found tied to the pier was big enough to carry only herself and one of the three possessions. She couldn't leave her dog on the bank with the little rabbit (the dog would frighten the poor rabbit), and she couldn't leave the rabbit alone with the carrots (the rabbit would eat all the carrots). But she still had to figure out how to cross the river safely with one possession at a time. How could she move back and forth across the river to get the three possessions safely to the other side?

LESSON

5.7

Proving Quadrilateral Properties

"For instance" is not a "proof."

JEWISH SAYING

Most of the paragraph proofs and flowchart proofs you have done so far have been set up for you to complete. Creating your own proofs requires a great deal of planning. One excellent planning strategy is "thinking backward." If you know where you are headed but are unsure where to start, start at the end of the problem and work your way back to the beginning one step at a time.

The firefighter below asks another firefighter to turn on one of the water hydrants. But which one? A mistake could mean disaster—a nozzle flying around loose under all that pressure. Which hydrant should the firefighter turn on?

Did you "think backward" to solve the puzzle? You'll find it a useful strategy as you write proofs.

To help plan a proof and visualize the flow of reasoning, you can make a flowchart. As you think backward through a proof, you draw a flowchart backward to show the steps in your thinking.

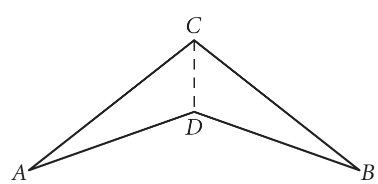
Work with a partner when you first try planning your geometry proof. Think backward to make your plan: start with the conclusion and reason back to the given. Let's look at an example.

A concave kite is sometimes called a **dart.**

EXAMPLE

Given: Dart $ADBC$ with $\overline{AC} \cong \overline{BC}$, $\overline{AD} \cong \overline{BD}$

Show: $\overline{CD}$ bisects $\angle ACB$



▸ Solution

Plan: Begin by drawing a diagram and marking the given information on it. Next, construct your proof by reasoning backward. Then convert this reasoning into a flowchart. Your flowchart should start with boxes containing the given information and end with what you are trying to demonstrate. The arrows indicate the flow of your logical argument. Your thinking might go something like this:

"I can show $\overline{CD}$ is the bisector of $\angle ACB$ if I can show $\angle ACD \cong \angle BCD$."

$\angle ACD \cong \angle BCD$ → $\overline{CD}$ is the bisector of $\angle ACB$

"I can show $\angle ACD \cong \angle BCD$ if they are corresponding angles in congruent triangles."

$\triangle ADC \cong \triangle BDC$ → $\angle ACD \cong \angle BCD$ → $\overline{CD}$ is the bisector of $\angle ACB$

"Can I show $\triangle ADC \cong \triangle BDC$? Yes, I can, by SSS, because it is given that $\overline{AC} \cong \overline{BC}$ and $\overline{AD} \cong \overline{BD}$, and $\overline{CD} \cong \overline{CD}$ because it is the same segment in both triangles."

$\overline{AD} \cong \overline{BD}$, $\overline{AC} \cong \overline{BC}$, $\overline{CD} \cong \overline{CD}$ → $\triangle ADC \cong \triangle BDC$ → $\angle ACD \cong \angle BCD$ → $\overline{CD}$ is the bisector of $\angle ACB$

By adding the reason for each statement below each box in your flowchart, you can make the flowchart into a complete flowchart proof.

1. $\overline{AD} \cong \overline{BD}$ — Given
2. $\overline{AC} \cong \overline{BC}$ — Given
3. $\overline{CD} \cong \overline{CD}$ — Same segment
4. $\triangle ADC \cong \triangle BDC$ — SSS
5. $\angle ACD \cong \angle BCD$ — CPCTC
6. $\overline{CD}$ is the bisector of $\angle ACB$ — Definition of angle bisector

Some students prefer to write their proofs in a flowchart format, and others prefer to write out their proof as an explanation in paragraph form. By reversing the reasoning in your plan, you can make the plan into a complete paragraph proof.

"It is given that $\overline{AC} \cong \overline{BC}$ and $\overline{AD} \cong \overline{BD}$. $\overline{CD} \cong \overline{CD}$ because it is the same segment in both triangles. So, $\triangle ADC \cong \triangle BDC$ by the SSS Congruence Conjecture. So, $\angle ACD \cong \angle BCD$ by the definition of congruent triangles (CPCTC). Therefore, by the definition of angle bisectors, $\overline{CD}$ is the bisector of $\angle ACB$. Q.E.D."

The abbreviation Q.E.D. at the end of a proof stands for the Latin phrase *quod erat demonstrandum,* meaning "which was to be demonstrated." You can also think of Q.E.D. as a short way of saying "Quite Elegantly Done" at the conclusion of your proof.

In the exercises you will prove some of the special properties of quadrilaterals discovered in this chapter.

EXERCISES

1. Let's start with a puzzle. Copy the 5-by-5 puzzle grid at right. Start at square 1 and end at square 100. You can move to an adjacent square horizontally, vertically, or diagonally if you can add, subtract, multiply, or divide the number in the square you occupy by 2 or 5 to get the number in that square.

   For example, if you happen to be in square 11, you could move to square 9 by subtracting 2 or to square 55 by multiplying by 5. When you find the path from 1 to 100, show it with arrows.

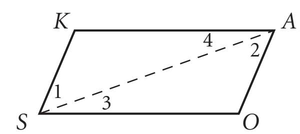
   Notice that in this puzzle you may start with different moves. You could start with 1 and go to 5. From 5 you could go to 10 or 3. Or you could start with 1 and go to 2. From 2 you could go to 4. Which route should you take? ⓗ

In Exercises 2–10, each conjecture has also been stated as a "given" and a "show." Any necessary auxiliary lines have been included. Complete a flowchart proof or write a paragraph proof.

2. Prove the conjecture: The diagonal of a parallelogram divides the parallelogram into two congruent triangles.

   Given: Parallelogram *SOAK* with diagonal $\overline{SA}$

   Show: $\triangle SOA \cong \triangle AKS$



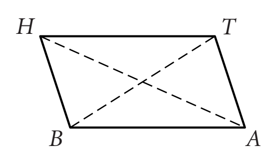
Flowchart Proof

1. *SOAK* is a parallelogram — Given
2. $\overline{SO} \parallel \overline{KA}$ — Definition of parallelogram
3. $\overline{OA} \parallel$ _?_ — _?_
4. $\angle 3 \cong \angle 4$ — AIA
5. $\angle 1 \cong \angle 2$ — _?_
6. $\overline{SA} \cong$ _?_ — _?_
7. $\triangle SOA \cong$ _?_ — _?_

3. Prove the conjecture: The opposite angles of a parallelogram are congruent.

   Given: Parallelogram *BATH* with diagonals $\overline{BT}$ and $\overline{HA}$

   Show: $\angle HBA \cong \angle ATH$ and $\angle BAT \cong \angle THB$



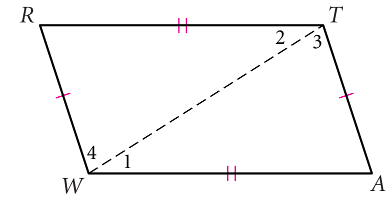
Flowchart Proof

1. Parallelogram *BATH* with diagonal $\overline{BT}$ — Given
2. $\triangle BAT \cong \triangle THB$ — Conjecture from Exercise 2
3. $\angle BAT \cong \angle THB$ — CPCTC
4. Parallelogram *BATH* with diagonal $\overline{HA}$ — _?_
5. $\triangle BAH = \triangle$ _?_ — _?_
6. $\angle HBA \cong \angle$ _?_ — _?_

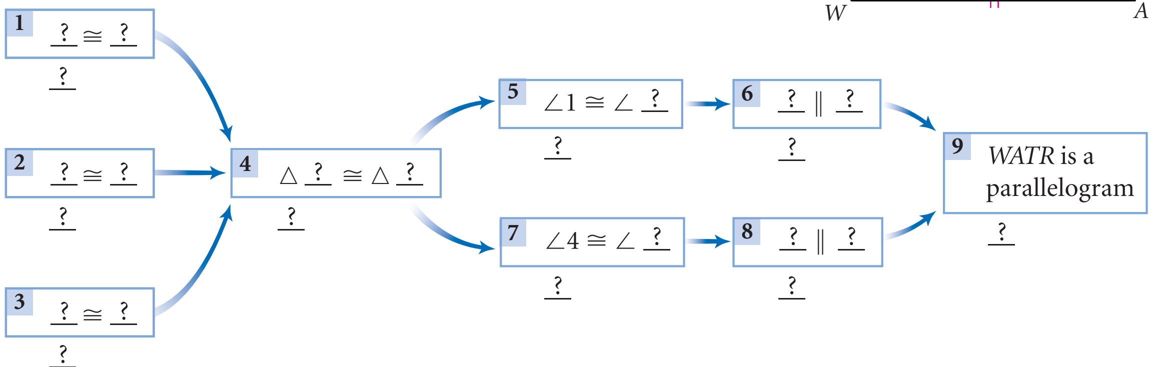
4. Prove the conjecture: If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: Quadrilateral $WATR$, with $\overline{WA} \cong \overline{RT}$ and $\overline{WR} \cong \overline{AT}$, and diagonal $\overline{WT}$

Show: $WATR$ is a parallelogram



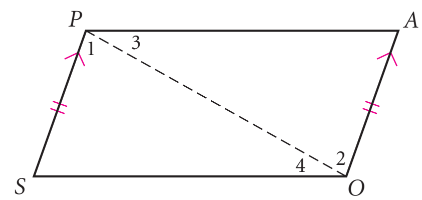
Flowchart Proof



5. Write a flowchart proof to demonstrate that quadrilateral $SOAP$ is a parallelogram.

Given: Quadrilateral $SOAP$ with $\overline{SP} \parallel \overline{OA}$ and $\overline{SP} \cong \overline{OA}$

Show: $SOAP$ is a parallelogram

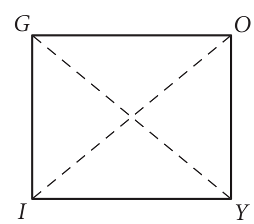


6. The results of the proof in Exercise 5 can now be stated as a proved conjecture. Complete this statement beneath your proof: "If one pair of opposite sides of a quadrilateral are both parallel and congruent, then the quadrilateral is a _?_."

7. Prove the conjecture: The diagonals of a rectangle are congruent. ⓗ

Given: Rectangle $YOGI$ with diagonals $\overline{YG}$ and $\overline{OI}$

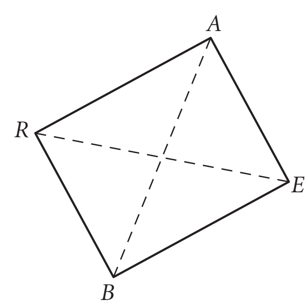
Show: $\overline{YG} \cong \overline{OI}$



8. Prove the conjecture: If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle. ⓗ

Given: Parallelogram $BEAR$, with diagonals $\overline{BA} \cong \overline{ER}$

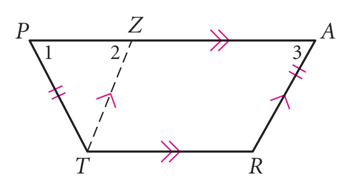
Show: $BEAR$ is a rectangle



9. Prove the Isosceles Trapezoid Conjecture: The base angles of an isosceles trapezoid are congruent.

Given: Isosceles trapezoid $PART$ with $\overline{PA} \parallel \overline{TR}$, $\overline{PT} \cong \overline{AR}$, and $\overline{TZ}$ constructed parallel to $\overline{RA}$

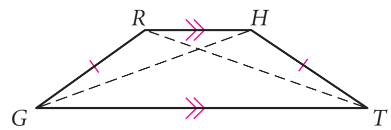
Show: $\angle TPA \cong \angle RAP$

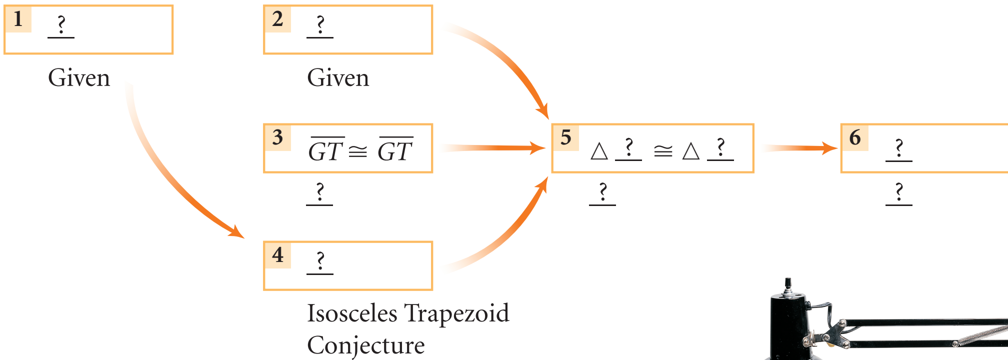


10. Prove the Isosceles Trapezoid Diagonals Conjecture: The diagonals of an isosceles trapezoid are congruent.

Given: Isosceles trapezoid $GTHR$ with $\overline{GR} \cong \overline{TH}$ and diagonals $\overline{GH}$ and $\overline{TR}$

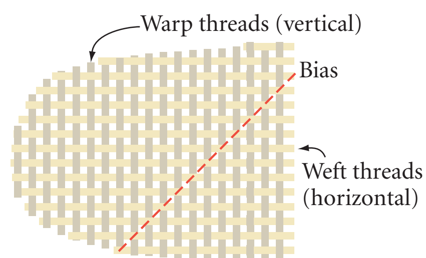
Show: $\overline{GH} \cong \overline{TR}$





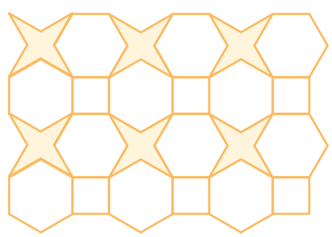
11. If an adjustable desk lamp, like the one at right, is adjusted by bending or straightening the metal arm, it will continue to shine straight down onto the desk. What property that you proved in the previous exercises explains why?

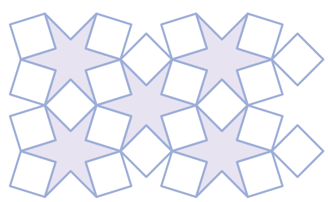
12. You have discovered that triangles are rigid but parallelograms are not. This property shows up in the making of fabric, which has warp threads and weft threads. Fabric is constructed by weaving thread at right angles, creating a grid of rectangles. What happens when you pull the fabric along the warp or weft? What happens when you pull the fabric along a diagonal (the *bias*)? ⓗ



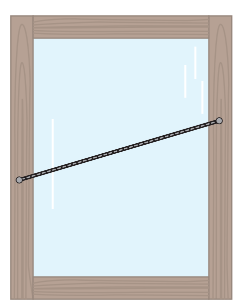
Review

13. Find the measure of the acute angles in the 4-pointed star in the Islamic tiling shown at right. The polygons are squares and regular hexagons. Find the measure of the acute angles in the 6-pointed star in the Islamic tiling on the far right. The 6-pointed star design is created by arranging six squares. Are the angles in both stars the same? ⓗ



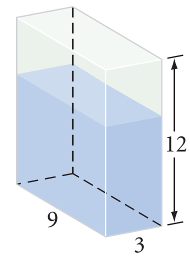


14. A contractor tacked one end of a string to each vertical edge of a window. He then handed a protractor to his apprentice and said, "Here, find out if the vertical edges are parallel." What should the apprentice do? No, he can't quit, he wants this job! Help him. ⓗ



15. The last bus stops at the school some time between 4:45 and 5:00. What is the probability that you will miss the bus if you arrive at the bus stop at 4:50? ⓗ

16. The 3-by-9-by-12-inch clear plastic sealed container shown is resting on its smallest face. It is partially filled with a liquid to a height of 8 inches. Sketch the container resting on its middle-sized face. What will be the height of the liquid in the container in this position? ⓗ



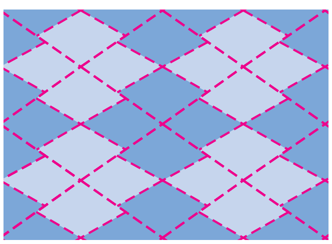
project

JAPANESE PUZZLE QUILTS

When experienced quilters first see Japanese puzzle quilts, they are often amazed (or puzzled?) because the straight rows of blocks so common to block quilts do not seem to exist. The sewing lines between apparent blocks seem jagged. At first glance, Japanese puzzle quilts look like American crazy quilts that must be handsewn and that take forever to make!

However, Japanese puzzle quilts do contain straight sewing lines. Study the Japanese puzzle quilt at right. Can you find the basic quilt block? What shape is it?

Mabry Benson designed this puzzle quilt, *Red and Blue Puzzle* (1994). Can you find any rhombic blocks that are the same? How many different types of fabric were used?



The puzzle quilt shown above is made of four different-color kites sewn into rhombuses. The rhombic blocks are sewn together with straight sewing lines as shown in the diagram at left. Look closely again at the puzzle quilt.

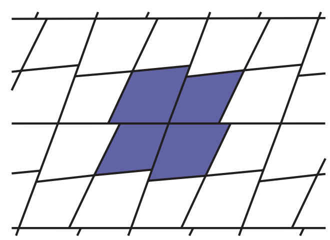
Now for your project. You will need copies of the Japanese puzzle quilt grid, color pencils or markers, and color paper or fabrics.

1. To produce the zigzag effect of a Japanese puzzle quilt, you need to avoid pseudoblocks of the same color sharing an edge. How many different colors or fabrics do you need in order to make a puzzle quilt?

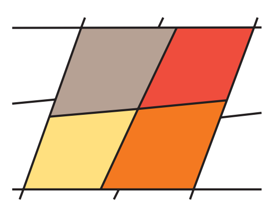
2. How many different types of rhombic blocks do you need for a four-color Japanese puzzle quilt? What if you want no two pseudoblocks of the same color to touch at either an edge or a vertex?

3. Can you create a four-color Japanese puzzle quilt that requires more than four different color combinations in the rhombic blocks?

4. Plan, design, and create a Japanese puzzle quilt out of paper or fabric, using the Japanese puzzle quilt technique.



Detail of a pseudoblock

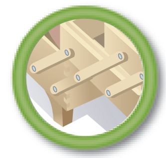


Detail of an actual block

CHAPTER 5 REVIEW

In this chapter you extended your knowledge of triangles to other polygons. You discovered the interior and exterior angle sums for all polygons. You investigated the midsegments of triangles and trapezoids and the properties of parallelograms. You learned what distinguishes various quadrilaterals and what properties apply to each class of quadrilaterals.

Along the way you practiced proving conjectures with flowcharts and paragraph proofs. Be sure you've added the new conjectures to your list. Include diagrams for clarity.



How has your knowledge of triangles helped you make discoveries about other polygons?

EXERCISES

You will need

Construction tools for Exercises **19–24**

1. How do you find the measure of one exterior angle of a regular polygon?

2. How can you find the number of sides of an equiangular polygon by measuring one of its interior angles? By measuring one of its exterior angles?

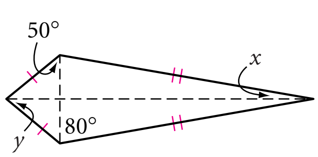
3. How do you construct a rhombus by using only a ruler or double-edged straightedge?

4. How do you bisect an angle by using only a ruler or double-edged straightedge?

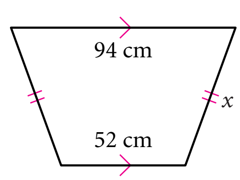
5. How can you use the Rectangle Diagonals Conjecture to determine if the corners of a room are right angles?

6. How can you use the Triangle Midsegment Conjecture to find a distance between two points that you can't measure directly?

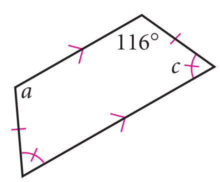
7. Find x and y.



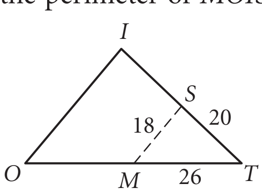
8. Perimeter $= 266$ cm. Find x.



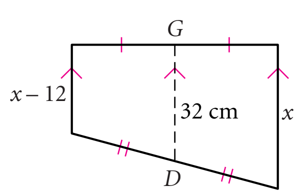
9. Find a and c.



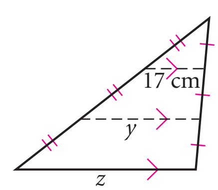
10. $\overline{MS}$ is a midsegment. Find the perimeter of $MOIS$.



11. Find x.



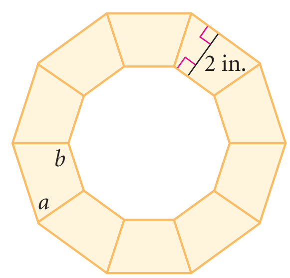
12. Find y and z.



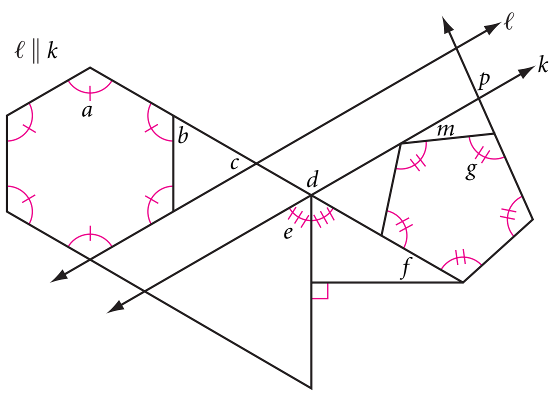
13. Copy and complete the table below by placing a yes (to mean always) or a no (to mean not always) in each empty space. Use what you know about special quadrilaterals.

	Kite	Isosceles trapezoid	Parallelogram	Rhombus	Rectangle
Opposite sides are parallel					
Opposite sides are congruent					
Opposite angles are congruent					
Diagonals bisect each other					
Diagonals are perpendicular					
Diagonals are congruent				No	
Exactly one line of symmetry	Yes				
Exactly two lines of symmetry					

14. **APPLICATION** A 2-inch-wide frame is to be built around the regular decagonal window shown. At what angles a and b should the corners of each piece be cut?



15. Find the measure of each lettered angle.

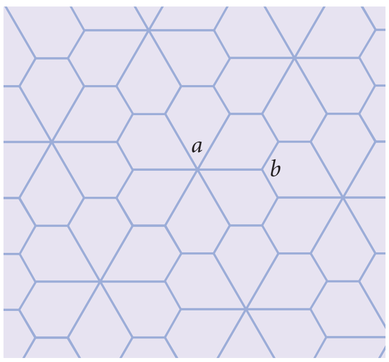


16. Archaeologist Ertha Diggs has uncovered one stone that appears to be a voussoir from a semicircular stone arch. On each isosceles trapezoidal face, the obtuse angles measure 96°. Assuming all the stones were identical, how many stones were in the original arch?

17. Kite *ABCD* has vertices $A(-3, -2)$, $B(2, -2)$, $C(3, 1)$, and $D(0, 2)$. Find the coordinates of the point of intersection of the diagonals.

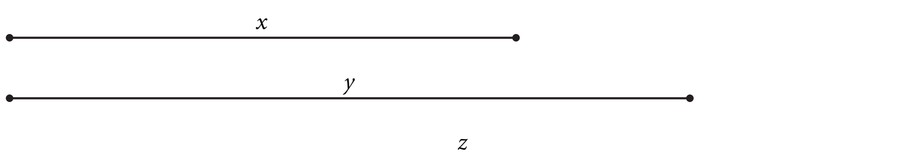
18. When you swing left to right on a swing, the seat stays parallel to the ground. Explain why.

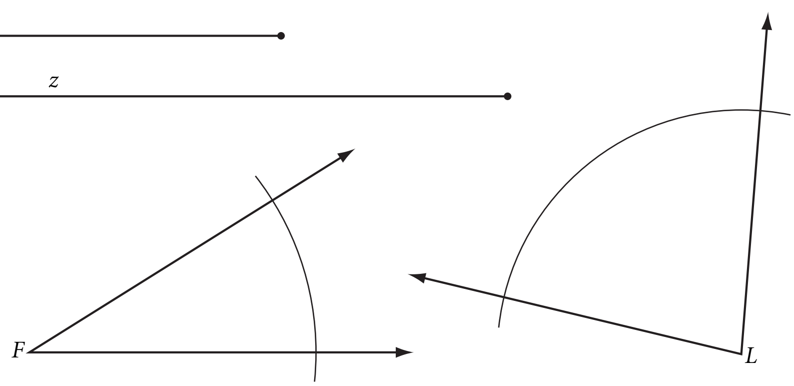
19. *Construction* The tiling of congruent pentagons shown below is created from a honeycomb grid (tiling of regular hexagons). What is the measure of each lettered angle? Re-create the design with compass and straightedge.



20. *Construction* An airplane is heading north at 900 km/hr. However, a 50 km/hr wind is blowing from the east. Use a ruler and a protractor to make a scale drawing of these vectors. Measure to find the approximate resultant velocity, both speed and direction (measured from north). ⓗ

Construction In Exercises 21–24, use the given segments and angles to construct each figure. Use either patty paper or a compass and a straightedge. The small letter above each segment represents the length of the segment.





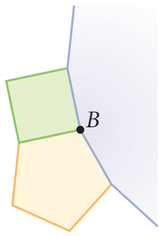
21. Construct rhombus *SQRE* with $SR = y$ and $QE = x$.

22. Construct kite *FLYR* given $\angle F$, $\angle L$, and $FL = x$.

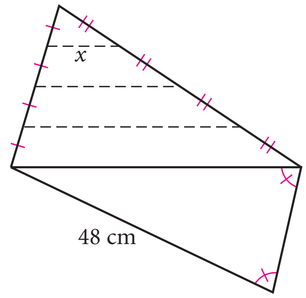
23. Given bases $\overline{LP}$ with length z and $\overline{EN}$ with length y, nonparallel side $\overline{LN}$ with length x, and $\angle L$, construct trapezoid *PENL*. ⓗ

24. Given $\angle F$, $FR = x$, and $YD = z$, construct two trapezoids *FRYD* that are not congruent to each other.

25. Three regular polygons meet at point B. Only four sides of the third polygon are visible. How many sides does this polygon have?



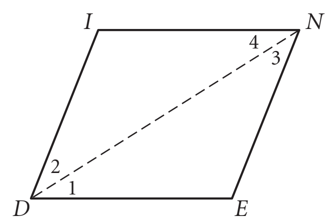
26. Find x.



27. Prove the conjecture: The diagonals of a rhombus bisect the angles.

Given: Rhombus $DENI$, with diagonal $\overline{DN}$

Show: Diagonal $\overline{DN}$ bisects $\angle D$ and $\angle N$



Flowchart Proof

1. $DENI$ is a rhombus — ?
2. $\overline{DE} \cong$? — ?
3. $\overline{NE} \cong$? — ?
4. $\overline{DN} \cong$? — ?
5. $\triangle$? $\cong \triangle$? — ?
6. $\angle 1 \cong \angle$?, $\angle 3 \cong \angle$? — ?
7. $\overline{DN}$ bisects $\angle IDE$ and $\angle INE$ — ?

TAKE ANOTHER LOOK

1. Draw several polygons that have four or more sides. In each, draw all the diagonals from one vertex. Explain how the Polygon Sum Conjecture follows logically from the Triangle Sum Conjecture. Does the Polygon Sum Conjecture apply to concave polygons?

2. A triangle on a sphere can have three right angles. Can you find a "rectangle" with four right angles on a sphere? Investigate the Polygon Sum Conjecture on a sphere. Explain how it is related to the Triangle Sum Conjecture on a sphere. Be sure to test your conjecture on polygons with the smallest and largest possible angle measures.

The small, precise polygons in the painting, *Boy With Birds* (1953, oil on canvas), by American artist David C. Driskell (b 1931), give it a look of stained glass.

3. Draw a polygon and one set of its exterior angles. Label the exterior angles. Cut out the exterior angles and arrange them all about a point. Explain how this activity demonstrates the Exterior Angle Sum Conjecture.

4. Is the Exterior Angle Sum Conjecture also true for concave polygons? Are the kite conjectures also true for darts (concave kites)? Choose your tools and investigate.

5. Investigate exterior angle sums for polygons on a sphere. Be sure to test polygons with the smallest and largest angle measures.

Assessing What You've Learned

GIVING A PRESENTATION

Giving a presentation is a powerful way to demonstrate your understanding of a topic. Presentation skills are also among the most useful skills you can develop in preparation for almost any career. The more practice you can get in school, the better.

Choose a topic to present to your class. There are a number of things you can do to make your presentation go smoothly.

- Work with a group. Make sure your group presentation involves all group members so that it's clear everyone contributed equally.
- Choose a topic that will be interesting to your audience.
- Prepare thoroughly. Make an outline of important points you plan to cover. Prepare visual aids—like posters, models, handouts, and overhead transparencies—ahead of time. Rehearse your presentation.
- Communicate clearly. Speak up loud and clear, and show your enthusiasm about your topic.

ORGANIZE YOUR NOTEBOOK Your conjecture list should be growing fast! Review your notebook to be sure it's complete and well organized. Write a one-page chapter summary.

WRITE IN YOUR JOURNAL Write an imaginary dialogue between your teacher and a parent or guardian about your performance and progress in geometry.

UPDATE YOUR PORTFOLIO Choose a piece that represents your best work from this chapter to add to your portfolio. Explain what it is and why you chose it.

PERFORMANCE ASSESSMENT While a classmate, a friend, a family member, or a teacher observes, carry out one of the investigations from this chapter. Explain what you're doing at each step, including how you arrived at the conjecture.